On the optimality of bargaining outcomes in the Collective-Particularistic multilateral bargaining game.

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Abstract

This note analyzes the efficiency properties of the equilibrium in a multilateral bargaining game in which a legislature divides a budget among collective and particularistic goods. We extend the model of Volden and Wiseman (2007) by considering

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smooth utility functions and consensus requirements ranging from simple-majority to unanimity. We show that when the private valuation of the private good is relatively high, only unanimity induces an (ex-ante) Pareto efficient outcome. Moreover, optimality can be easily attained by using sequential negotiations, independently of the majority requirement.

Keywords: Non-cooperative bargaining, quota rules.

1 Introduction

Volden and Wiseman (2007) consider a committee of n players with linear preferences that are posed to allocate a fixed budget between collective and particularistic spending.[1] Negotiations proceed over time through a random proposers alternating bargaining game, and decisions require the acceptance of a simple majority of players. In this note, we extend their model to (i) quasi-linear preferences and (ii) consensus requirements that range from simple majority to unanimity. We characterize the stationary subgame perfect equilibria (which, we show must be symmetric) for any consensus requirement, and analyze the efficiency properties of these equilibria.

To our view, this slight modification of Volden and Wiseman (2007) contributes to a better understanding of the patterns driving the bargaining outcomes in the collective-particularistic model: When q players are required for a proposal to succeed, the equilib-

[1] We also refer to Volden and Wiseman (2008) where the cutpoints specified in their previous work are corrected.

rium provision of collective good remains between what would be optimal from an individual point of view and what would be optimal for a society of exactly q agents. Hence, if preferences are smooth and $q < n$ then ex-ante Pareto optimality is never achieved unless the relative valuation of the collective good is high enough so that all budget is assigned to the collective good. This contrasts with Volden and Wiseman (2007) where ex-ante Pareto efficiency is attained also for large valuations of the particularistic good. This is due to their assumption that collective and particularistic goods are perfect substitutes, which implies that the Pareto optimal spending on collective good may coincide with the optimal provision for a coalition of half of the members of the committee. In addition, unlike Volden and Wiseman (2007), we find that collective spending can coexist with a minimal-winning coalition of committee members receiving particularistic goods.

The model considered stands in the branch of the literature that starts with Baron and Ferejohn (1989), addressing collective choices as a process where bargaining and voting are combined. This analysis has been extended by Banks and Duggan (2000, 2004) that provide a general analysis of existence of equilibrium for any convex set of policies and concave utilities. In particular, Banks and Duggan (2000) covers the model where a committee negotiates over both collective and particularistic spending. Some papers (e.g., Austen-Smith and Banks 1988, Baron and Diermeier 2001, Crombez 1996, Jackson and Moselle 2002) analyze this particular environment without imposing any trade-off between the collective good choice and the size of private transfers. Contrarily, in our extension of Volden and Wiseman (2007) the size of the collective good is related to private transfers through a budget constraint. This is also the approach of Jackson (2011) in a simple majority 3-player bargaining game. He characterizes the equilibrium when the players are symmetric and shows that equilibrium proposals randomize amongst all legislators only if the game is symmetric. We refer the reader to Volden and Wiseman (2007) for a detailed discussion of the literature on these collective-particularistic bargaining models.

In the next section, we present the model and some preliminary properties of the equilibrium. Section 3 characterizes the unique equilibrium and discusses its efficiency properties. In Section 4, we discuss our findings and show that ex-ante Pareto optimality can be attained by using sequential negotiations.

2 The model

A set of n players, say N, must allocate a fixed budget, which is normalized to one, among a collective and n particularistic goods. Specifically, players must agree on an allocation $\mathbf{x} \in X = \left\{(x_1, ..., x_n, y) \in \mathbb{R}_+^{n+1} : \sum_{l=1}^{n} x_l + y \leq 1\right\}$, where x_i denotes the amount of private good that player i receives and y is the amount spent on the collective good. Agreements are reached through a negotiation process among agents that proceeds over discrete time as follows: at each period $t \geq 0$ one player $j \in N$ is randomly selected as the proposer, with each player having the same probability of being selected. Then, she proposes an allocation of the surplus $\mathbf{x}^j \in X$. The proposal is approved if at least $q-1$ other agents accept it, so q is the required quota or the minimum size of the winning coalition. This quota ranges from simple majority to unanimity, *i.e.* $q \in \{\lfloor n/2 \rfloor + 1, ..., n\}$, where $\lfloor z \rfloor$ denotes the integer part of z. In case of approval, the proposed allocation is implemented and the game ends; otherwise the game moves to period $t+1$ where a new player is randomly selected to make a proposal, and so on.

Particularistic goods benefit players privately whereas the collective good spending benefits all players. Upon agreement on $\mathbf{x} \in X$ at period t player $j \in N$ obtains utility $\delta^t u_j(\mathbf{x})$, where

$$u_j(\mathbf{x}) = \alpha x_j + y^{\beta},$$

$\beta \in (0,1)$ and $\alpha > 0$.

A *strategy* for a given player consists on a proposal and an acceptance/rejection rule for each subgame. The strategy is *stationary* if both the proposal and the acceptance rule are independent of previous rejected proposals. A *stationary subgame perfect equilibrium* (henceforth, *SSPE*) is a profile of stationary strategies that are mutually best responses in each subgame.

Any SSPE induces a profile of expected utilities $(\overline{u}_1, ..., \overline{u}_n)$, which by stationarity remains constant after any possible rejection. Thus, in any SSPE, any agent i is characterized by her acceptance set $A_i = \left\{\mathbf{x} \in \mathbb{R}_+^{n+1} : u_j(\mathbf{x}) \geq \delta \overline{u}_j\right\}$; i.e., a set of proposals that she would accept. Likewise, a proposal x is accepted whenever $x \in A(q) = \cap_{\ j \in W(q)} A_j$ where $W(q) = \{S \subset N : |S| \geq q\}$ denotes the set of winning coalitions.

Some properties of the SSPE are specified below. In particular, the next two results show that any agreement is reached immediately and that any SSPE is symmetric.

Lemma 1 *In any SSPE there is no delay.*

Proof. Consider an SSPE yielding $(\overline{u}_1, ..., \overline{u}_n)$. Note that $A(q) \neq \varnothing$ since otherwise perpetual disagreement is inconsistent $A(q) \neq \varnothing$. Player j prefers to delay the agreement iff there is no $\mathbf{x} \in A(q)$ such that $u_j(\mathbf{x}) \geq \delta\overline{u}_j$. However, this implies that, $\overline{u}_j \leq \delta\overline{u}_j$ which is a contradiction if $\overline{u}_j > 0$. Moreover, if $\overline{u}_j = 0$ is inconsistent. To see this, note that $\overline{u}_j = 0 \Longleftrightarrow y = 0$. Thus $\sum \overline{u}_j < \alpha$, implying that j can obtain $x_i = 1 - \left(\sum \overline{u}_j / \alpha\right) > 0$ when proposing. ■

Lemma 2 *Any SSPE yields the same expected utility for all agents.*

Proof. See the Appendix. ■

Hence, the following is immediate.

Corollary 1 *In any SSPE, the particularistic share received by any responder in a winning coalition is the same.*

In what follows we take advantage of this symmetry to simplify notation. A proposal $\mathbf{x}^j \in X$ can be denoted as a triple $(x_j^j, x_i^j; y^j)$, where x_j^j is the part of the surplus that player j proposes to keep for herself, x_i^j is the particularistic share that j gives to the other members of the winning coalition and y^j denotes the collective spending.

Before characterizing the SSPE, some comments about the meaning of Pareto optimality in our environment are worth. For any $s \in \{1, ..., n\}$, let y_s denote the allocation of collective good that maximizes $\alpha(1-y) + sy^{\beta}$, which is given by $y_s = (\beta s/\alpha)^{\frac{1}{1-\beta}}$. That is, y_s denotes the unique Pareto optimal provision of collective good in an economy with s agents and no restrictions in the particularistic payments of these agents. In our model there is a fixed budget to share and negative payoffs are not allowed. Thus, the concept of Pareto optimality is restricted. In particular, any bargaining outcome $(1-y, 0, y)$, with $y \in [y_1, y_n]$ is ex-post Pareto optimal. Thus, not surprisingly, the bargaining outcome is always ex-post Pareto efficient. However, when α is high (relatively to δ), all agents would obtain a positive particularistic expected share. Thus, ex-ante efficiency would require that $y^* = y_n$, which can be attained only if $q = n$.

3 Results

For any $\beta \in (0,1)$ and any minimum winning coalition size q, the SSPE proposals are characterized below. First, we introduce the function f, and its properties, that will be used in the following.

Lemma 3 *Let* $f(y, \alpha; \delta) = \frac{\alpha\delta}{n}(1-y) - (1-\delta)y^{\beta}$. *Then,* $f(y; \alpha, \delta) = 0$ *has a unique solution* y^*, *which is (i) decreasing in* α *and (ii) increasing in* δ.

Proof. Immediate. ■

Proposition 1 *The SSPE proposals are:*

(i) $\mathbf{x}^* = (0, 0, 1)$ *if* $\alpha < \beta$,

(ii) $\mathbf{x}^* = (1 - y_1, 0, y_1)$ *if* $\alpha \in [\beta, \alpha_1)$ *where* α_1 *solves* $f(y_1, \alpha_1; \delta) = 0$,

(iii) $\mathbf{x}^* = (1 - \widehat{y}, 0, \widehat{y})$ *if* $\alpha \in [\alpha_1, \alpha_2)$ *where* $f(\widehat{y}, \alpha; \delta) = 0$ *and* $f(y_q, \alpha_2; \delta) = 0$,

(iv) $\mathbf{x}^* = (1 - (q-1)x - y_q, x, y_q)$ *if* $\alpha \geq \alpha_2$ *where* $x = \frac{1}{\alpha} f(y_q, \alpha; \delta)$.

Proof. Partial derivatives of u_i with respect to x_i and y reveal that collective spending is preferred to particularistic spending whenever $y < y_1$. If $\alpha < \beta$ the threshold y_1 cannot be reached and therefore $\mathbf{x} = (0, 0, 1)$ gains unanimous support.

When $\alpha \geq \beta$ the best outcome for any player is $\bar{\mathbf{x}} = (1 - y_1, 0, y_1)$. Moreover, this allocation gains unanimous support whenever

$$\left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{1-\beta}} \geq \frac{\delta\alpha}{n}\left(1 - \left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\beta}}\right) + \delta\left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{1-\beta}};$$

i.e., when responders weakly prefer to accept the proposal rather than delaying the agreement one period. This condition can be written as:

$$f(y_1, \alpha; \delta) \leq 0.$$

Since f increases in α, this inequality holds whenever $\alpha \leq \alpha_1$, where α_1 satisfies $f(y_1, \alpha_1; \delta) = 0$. Hence, when $\alpha \in [\beta, \alpha_1)$ the proposer can obtain $\mathbf{x} = \bar{\mathbf{x}}$.

When $\alpha > \alpha_1$ the proposer can no longer obtain $\bar{\mathbf{x}}$. In order to get the support of (at least) $q - 1$ players, she will have to either (a) increase the provision of collective good or (b) give a positive share of the surplus in terms of particularistic good to the rest of the winning coalition members. We next claim that alternative (a) is preferred to (b) iff $y^i < y_q = (\beta q/\alpha)^{\frac{1}{1-\beta}}$. To prove this claim, let $\mathbf{x}^i = (1 - (q-1)x^i - y^i, x^i, y^i)$ denote the proposal of agent i. Notice that $\mathbf{x}^i$ generates utilities:

$$u_i = \alpha(1 - y^i - (q-1)x^i) + \left(y^i\right)^\beta$$

and

$$u_j = \alpha x^i + \left(y^i\right)^\beta$$

where j is a typical member of the winning coalition different from i. The proposer i may increase u_j using alternatives (a) and (b).

(a) $dy^i > 0$ and $dx^i = 0$. This implies $dy^i = \frac{\left(y^i\right)^{1-\beta}}{\beta} du_j$ and $du_i = \left(1 - \frac{\alpha}{\beta}\left(y^i\right)^{1-\beta}\right) du_j$.

(b) $dy^i = 0$ and $dx^i > 0$. This implies $dx^i = \frac{du_j}{\alpha}$ and $du_i = (1-q)du_j$.

Thus, player i prefers alternative (a) to (b) whenever $\left(1 - \frac{\alpha}{\beta}\left(y^i\right)^{1-\beta}\right) du_j > (1-q)du_j$. I.e., when $y^i < y_q$.

Therefore, players propose $(1-y, 0, y)$ whenever $y < y_q$. This allocation gains unanimous support whenever

$$f(y, \alpha; \delta) \leq 0$$

To maximize her utility the proposer sets $y = \widehat{y}$ such that $f(\widehat{y}, \alpha; \delta) = 0$. Since $\frac{\partial \widehat{y}}{\partial \alpha} > 0$, $\widehat{y} < y_q$ if and only if $\alpha \leq \alpha_2$ where $f(y_q, \alpha_2; \delta) = 0$. Therefore, the proposal $(1-\widehat{y}, 0, \widehat{y})$ gains unanimous support whenever $\alpha \in [\alpha_1, \alpha_2)$.

When $\alpha > \alpha_2$, $(1-y_q, 0, y_q)$ is not enough to receive the approval of a winning coalition. Thus, case (b) applies: the agent prefers to increase the utility of the responders by increasing their particularistic share. I.e., players will propose $(1-(q-1)x - y_q, x, y_q)$ where x is set to get the support of those non-proposers who are included in the winning coalition. I.e., x adjusts to make responders indifferent between accepting the proposal and delaying the agreement one period. I.e.,

$$\alpha x + y_q^\beta = \frac{\delta}{n}\alpha\left(1 - y_q\right) + \delta y_q^\beta,$$

yielding $x = f\left(y_q, \alpha; \delta\right)/\alpha$. ■

As an illustration, in the next example the SSPE collective spending is explicitly calculated.

Example 1 *Let $n = 7$, $\beta = 0.5$, $q = 4$, and $\delta = 0.9$. The next figure depicts the SSPE collective spending (dashed line) and the ex-ante Pareto efficient provision of collective good as a function of α.*

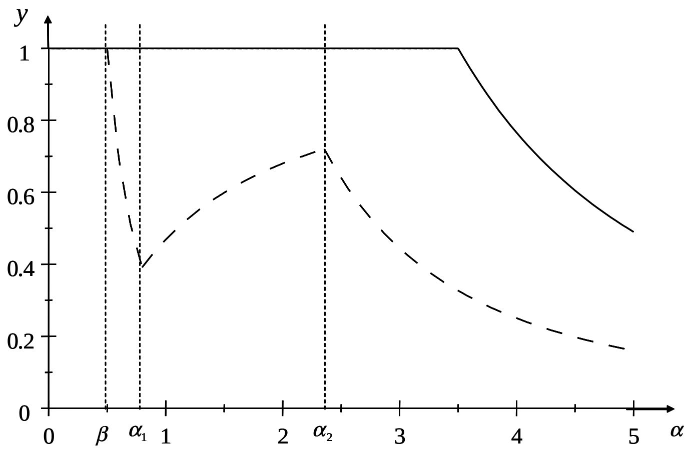

Figure 1

For low values of α, the proposer can obtain her first best allocation $(1 - y, 0, y)$, where $y = \min\{y_1, 1\}$. However, when $\alpha > \alpha_1$, her first best does not suffice to obtain the support of a winning coalition. I.e., large compensations to responders are required.

For moderate values of α ($\alpha \in [\alpha_1, \alpha_2]$), collective good provision is used to gain the support of a winning coalition. In fact, the support of all members will be gained within this range. Moreover, as the equilibrium expected utilities of the agents increase with α, the collective spending must also increase as α grows.[2] Yet, when $\alpha > \alpha_2$ the collective share required to obtain the support of $q-1$ responders would exceed y_q, i.e. the optimal provision of collective good for a coalition of q players. In these cases, the proposer will combine particularistic shares with this "optimal" provision of collective good. Moreover, since y_q decreases in α, the SSPE collective spending also decreases in α when $\alpha > \alpha_2$.

In our model, the SSPE collective good spending is always positive, as $\lim_{y \to 0} \frac{\partial u_i}{\partial y} = \infty$. This contrasts with the linear model, specially for a large valuation of the particularistic good, where the proposers build the minimal winning coalition by combining the amount $y = y_q$ of collective good (which decreases with α) with particularistic shares. Moreover, besides the fact that SSPE collective spending is continuous in α, this fact has also strong implications on the ex-ante efficiency properties of the SSPE outcomes. While in the linear model efficiency is also attained for large values of α, this can never happen when preferences are smooth unless unanimity is required.

Corollary 2 *If $q < n$ then ex-ante Pareto efficiency can be obtained only if SSPE proposals are* $(0,0,1)$*. Moreover, this happens whenever*

1. $\alpha \leq \beta$ *if* $\delta < 1$, *or*

[2] This counter-intuitive result that more is spent on collective good when its relative valuation decreases is also remarked by Volden and Wiseman (2007).

2. $\alpha \leq q\beta$ *when* $\delta \to 1$*.*

Proof. It is obvious that when $\alpha \leq \beta$ all agents will propose $(0,0,1)$, which is (ex-ante) Pareto optimal. Moreover, it can be easily checked (from the previous proposition) that when $\alpha \in [\alpha_1, \alpha_2)$ and $\delta < 1$ then the collective share is always smaller than 1, which would be the optimal collective spending.

When $\delta \to 1$, from the previous proposition, we obtain $\alpha_1 = \beta$, $\alpha_2 = q\beta$ and $y = 1$ whenever $\alpha \in [\beta, q\beta)$, which is ex-ante Pareto efficient. Instead, as y_q does never coincide with y_n when $q < n$, we have that the SSPE allocation is always ex-ante Pareto inefficient when $\alpha > \alpha_2$, independently of the existence or not of transaction costs. ■

The next example illustrates how δ affects the collective spending: a reduction of the transaction costs may restore ex-ante efficiency only for moderate values of α.

Example 2 *Consider* $n = 7$*,* $\beta = 0.5$*,* $q = 4$*. The next picture plots the SSPE collective good provision for different values of* δ*. Dashes, circles and crosses represent the collective good levels for* $\delta = 0.9$*,* $\delta = 0.95$*, and* $\delta = 0.99$*, respectively. Solid line represents the* *optimal provision. Note that δ affects the collective share only when $\alpha \in [\alpha_1, \alpha_2)$.*

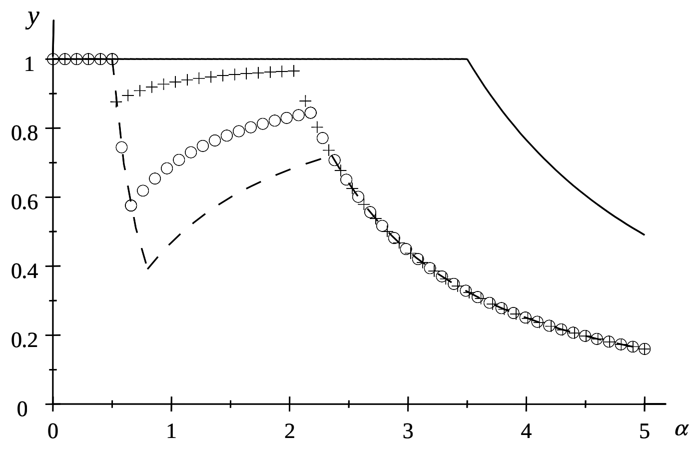

Figure 2

As proposals are unanimously accepted when $\alpha \in [\alpha_1, \alpha_2)$, one can think that the winning coalition size q has no effects on the equilibrium outcome. However, this is not exact, since α_2 is a function of q. Moreover, when $\alpha \geq \alpha_2$ the minimal winning coalition size q becomes crucial in determining the size of the collective share, as in these cases it coincides with the optimal provision in a society of exactly q players, which increases in q.

Corollary 3 *Any increase in q induces to a higher α_2 and to a larger SSPE collective spending.*

The next example illustrates the point.

Example 3 *Let $n = 7$, $\beta = 0.5$, $\delta = 0.9$. Next figure depicts the SSPE collective good provision for different values of q. Specifically, dashes, circles, crosses, and diamonds* *represent the equilibrium provision of collective good with $q = 4$, $q = 5$, $q = 6$, and $q = 7$, respectively. Solid line represents the optimal provision.*

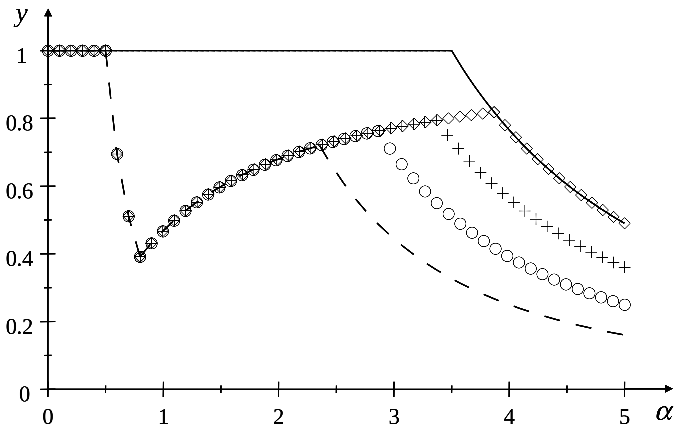

Figure 3

4 Discussion

In the linear model, there is efficient provision of collective good either (i) for low values of α, where $y = 1$ or (ii) for $\alpha > n\beta$, where efficiency requires $y = 0$. However, in our extension to quasi-linear preferences, unless unanimity is required, ex-ante efficiency is attained only for low values of α.

Inefficiency stems from two advantages that the bargaining game grants to the proposer. First, the proposer takes advantage of the impatience of others and proposes the level of collective good which is closest to her best that guarantees acceptance; and second, only $q - 1$ are required for an agreement. Reducing the impatience of the players would force the proposer to increase the collective good spending, thus increasing (ex-ante) efficiency. However, as the proposer will never provide more collective good than the optimal level for a society of q agents, Pareto efficiency cannot be obtained unless $q = n$ and α is large enough (relative to δ).

It is worth to note that, the restriction of the policy space to positive shares is not determinant for the inefficiency result. By adding a participation constraint (to avoid expropriation of the minority) and allowing for negative shares, the proposer could extract some positive rents only from $q - 1$ agents. The intuition is clear: An agent would reject a proposal with a negative particularistic share if she anticipates that the proposal is accepted without her approval. Thus, free riding behavior would make impossible to obtain positive rents from more than $q - 1$ players. The only thing that will change with respect to the previous results is that (i) the collective share would be always y_q, irrespective of α, and (ii) the members in the winning coalitions would obtain a negative particularistic share when $\alpha < \alpha_2$. Hence, under non-unanimous consent, the inefficiency would remain independently of both transaction costs and the restriction of proposals to positive particularistic shares.

In order to obtain Pareto efficiency, agents must internalize the social benefits of collective spending. This can be done by using sequential negotiations. Consider the situation where agents decide first the part of the budget assigned to the collective good. Once they agree, then the remaining surplus is distributed among the committee members. In each stage, decisions are taken through the alternating proposal multilateral bargaining game with random proposers. Since agents are symmetric, they all have the same preferences over the size of the collective spending in the first stage. Moreover, it coincides with the Pareto optimal level, independently of the consensus requirement. Thus, because of symmetry, issue-by-issue is ex-ante preferred by all agents.[3]

Proposition 2 *Under issue-by-issue bargaining, efficiency is restored when the level of collective good is selected first.*

Proof. For any $y \in [0,1]$, the expected utility of any player is in the ensuing bargaining game is $y^{\beta} + \alpha (1-y)/n$. This implies that all agents, unanimously prefer $y = \min \left\{ \left(\frac{n\beta}{\alpha}\right)^{\frac{1}{1-\beta}}, 1 \right\}$. ■

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[3] Notice that the situation where agents negotiate first on the private spending is equivalent to the base model.

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Proof of Lemma 2.

Let $\overline{u}_1 \leq ....\overline{u}_q \leq ... \leq \overline{u}_n$ with some strict inequality. Expected utilities are given by

$$n\overline{u}_1 = \delta\left[\sum\nolimits_{i \notin \{1,n\}} \overline{u}_1^i + \overline{u}_1^1 + \overline{u}_1^n\right]$$
$$n\overline{u}_n = \delta\left[\sum\nolimits_{i \notin \{1,n\}} \overline{u}_n^i + \overline{u}_n^1 + \overline{u}_n^n\right]$$

where $\overline{u}_j^i$ denotes the expected utility obtained by player j when i is the proposer.

Since the cheapest winning coalition is chosen by the proposer, then $u_1^j \geq \delta\overline{u}_1$ if $u_n^j = \delta\overline{u}_n$ and $u_1^j \geq u_n^j$ otherwise. Thus,

$$n\left[\overline{u}_n - \overline{u}_1\right] \leq m\delta\left[\overline{u}_n - \overline{u}_1\right] + \delta\left[\overline{u}_n^1 + \overline{u}_n^n - \overline{u}_1^1 - \overline{u}_1^n\right]$$

for some $m \in [0, n-2]$.

Moreover, either $u_1^h \geq \delta\overline{u}_1$ and $u_h^l \leq \delta\overline{u}_n$ or 1 is not included in the cheapest winning coalition. In the last case, we have that the optimal proposal of agent 1 is also z^h. Thus,

$$[n - m\delta]\left[\overline{u}_n - \overline{u}_1\right] \leq 0,$$

which is a contradiction.

In case that $u_1^h \geq \delta\overline{u}_1$ and $u_h^l \leq \delta\overline{u}_n$ we obtain

$$[n - (m+1)\,\delta]\left[\overline{u}_n - \overline{u}_1\right] \leq \delta\left[\overline{u}_n^n - \overline{u}_1^1\right].$$

By mimicking the proposal of agent n player 1 can obtain

$$\overline{u}_1^1 \geq a\left(1 - y_n - C - x_n^1\right) + y_n^\beta$$

where $x_h^l = \max\left\{\frac{\delta\overline{u}_h - y_h^\beta}{a}, 0\right\}$. Thus, $\overline{u}_n^n - \overline{u}_1^1 = a\left(x_n^1 - x_1^n\right)$. We distinguish three cases:

1. $x_n^1 = 0$, which implies $x_1^n = 0$. In this case, we obtain

$$[n - (m+1)\,\delta]\,[\overline{u}_n - \overline{u}_1] \leq 0,$$

which is a contradiction.

2. $x_n^1 = \frac{\delta \overline{u}_n - y_n^\beta}{a}$ and $x_1^n = 0$. Since $y_n^\beta \geq \delta \overline{u}_1$,

$$a\left(x_n^1 - x_1^n\right) = a x_n^1 = \delta \overline{u}_n - y_n^\beta \leq \delta\left(\overline{u}_n - \overline{u}_1\right)$$

and therefore

$$[n - (m+2)\,\delta]\,[\overline{u}_n - \overline{u}_1] \leq 0,$$

which is also a contradiction.

3. Similarly, if $x_n^1 = \frac{\delta \overline{u}_n - y_n^\beta}{a}$ and $x_1^n = \frac{\delta \overline{u}_1 - y_1^\beta}{a}$ then $a\left(x_n^1 - x_1^n\right) = \delta\left(\overline{u}_n - \overline{u}_1\right)$ and a contradiction is obtained, too.

■