HEGY Tests in the Presence of Moving Averages*

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Abstract

We analyze the asymptotic distributions associated with the seasonal unit root tests of the Hylleberg et al. (1990) procedure for quarterly data when the innovations follow a moving average process. Although both the t- and F-type tests suffer from scale and shift effects compared with the presumed null distributions when a fixed order of autoregressive augmentation is applied, these effects disappear when the order of augmentation is sufficiently large. However, as found by Burridge and Taylor (2001) for the autoregressive case, individual t-ratio tests at the the semi-annual frequency are not pivotal even with high orders of augmentation, although the corresponding joint F-type statistic is pivotal. Monte Carlo simulations verify the importance of the order of augmentation for finite samples generated by seasonally integrated moving average processes.

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1 Introduction

The predominant approach to testing for unit roots in the context of seasonal time series is that of Hylleberg, Engle, Granger and Yoo [HEGY] (1990), who develop the approach of Dickey and Fuller (1979) in this context. More specifically, HEGY provide tests for unit roots at the zero and each seasonal frequency, within the overall null hypothesis that seasonal (or annual) differencing is required to induce stationarity in a quarterly time series. In common with Dickey and Fuller (1979), HEGY assume that the process has an autoregressive (AR) form, and hence AR augmentation is used to account for serial correlation. However, Burridge and Taylor (2001) show that even with appropriate augmentation, AR innovations cause the HEGY t-ratio statistics associated with the complex conjugate unit roots at the semi-annual frequency $\pi/2$ to depend on nuisance parameters. Nevertheless, the distributions of the t-ratio tests associated with frequencies zero and π, together with the joint F-type test associated with frequency $\pi/2$, remain pivotal in this situation.

One implication of the analysis of Burridge and Taylor (2001) is that the consequences of serial correlation need to be considered carefully when applying HEGY seasonal unit root tests, since results that apply for the (zero frequency) Dickey-Fuller test do not necessarily carry over to the seasonal frequencies. While sometimes recognising that serial correlation is not necessarily of an AR form, researchers applying HEGY tests simply assume that any moving average (MA) component can be approximated through AR augmentation of a sufficiently high order, implying that the results of Said and Dickey (1984) carry over to this case. Indeed, such empirical analyses typically employ some data-dependent technique for specifying the appropriate AR lag order. To our knowledge, no theoretical analysis underpins the conditions under which this approach will deliver the assumed asymptotic distributions of the seasonal unit root tests in the presence of MA autocorrelation.

In the conventional (zero frequency) unit root context, the Monte Carlo studies of Schwert (1989) and Agiakloglou and Newbold (1992) find that the Augmented Dickey-Fuller (ADF) test can suffer from large size distortions when the true innovations are of an MA form. In this light, Galbraith and Zinde-Walsh (1999) obtain the asymptotic distribution of the ADF t-ratio test, showing that an AR augmentation of order $p = O(\delta \ln T)$, with $\delta > 0$ and T being the sample size, is required for negligible asymptotic size distortions to exist in the presence of MA innovations.

Although there has been little analysis of the impact of MA disturbances on HEGY tests, the Monte Carlo study of Rodrigues and Osborn (1999) finds that substantial size distortions can occur in the presence of a simple seasonal MA, with the extent of distortion depending on the MA coefficient and the augmentation order. The present paper provides the theoretical rationale for these results, by analyzing the asymptotic distributions of the HEGY seasonal unit root in the presence of MA disturbances. In particular, and analogously with the (nonseasonal) results of Galbraith and Zinde-Walsh (1999), the required number of lags p of the seasonal difference of the time series required to ensure convergence to the presumed asymptotic distribution is such that $p = O(\delta \ln N)$ with $\delta > 0$ where N is the number of years and (in the quarterly case) the total sample size is $4N$. We also show that the non-pivotal nature of the distributions of the t-ratio statistics uncovered by Burridge and Taylor (2001) at the annual frequency for an autoregressive process continue to apply in the MA case, with the F-type statistic being pivotal (with sufficient augmentation).

To be specific, we obtain the asymptotic distributions of the HEGY seasonal unit root tests in the presence of MA innovations for a given order of AR augmentation p and also as the order of augmentations grows. The analysis here follows similar lines to del Barrio Castro and Osborn (2008), although the focus of interest in that study is the distribution of the HEGY test statistics when the true process is of a periodic integrated form. The organisation of the present paper is as follows. The next section presents the preliminaries needed for the subsequent analysis, after which (in Section 3) we present our principal results. A Monte Carlo study of the distribution of the HEGY statistics in the presence of MA innovations follows in Section 4, with Section 5 concluding. An Appendix provides the proof of the analytical results of Section 3.

2 Preliminaries

Consider the following quarterly univariate seasonally integrated (SI) process with MA disturbances:

$$x_{s\tau} = x_{s,\tau-1} + u_{s\tau}, \quad s = 1,2,3,4, \tau = 1,2,\cdots,N \tag{1}$$

$$\text{with} \quad u_{s\tau} = \theta(L)\,\epsilon_{s\tau}, \quad \theta(L) = (1 + \theta_1 L + \cdots + \theta_q L^q)$$

where for observation $x_{s\tau}$ the first subscript refers to the season (s) and the second subscript to the year (τ). When $s = 1$, it is understood that $x_{s-1,\tau} = x_{4,\tau-1}$. For ease of presentation, we assume that observations are available for precisely N years, with total sample size therefore $T = 4N$, and starting values $x_{s\tau} = 0$ ($s = 1, 2, 3, 4$). Clearly, the seasonal random walk is rendered stationary through the use of the seasonal difference operator $\Delta_4 = \left(1 - L^4\right)$ (where L is the quarterly lag operator, so that $L^k x_{s\tau} = x_{s-k,\tau}$ and $L^4 x_{s\tau} = x_{s,\tau-1}$). The error process in (1) follows an invertible MA(q) process, so that the roots of $\theta(z) = 1 + \sum_{i=1}^{q} \theta_i z^i$ all lie strictly outside the unit circle. Finally, the innovations $\{\epsilon_{s\tau}\}$ form a martingale difference sequence (MDS) with constant conditional variance σ^2; see Fuller (1996, Theorem 5.3.3) for details.

As is well known, the MA(q) process $u_{s\tau}$ in (1) has autocovariances

$$\gamma_k = E[u_{s\tau} u_{s-k,\tau}] = \begin{cases} \sigma^2 \sum_{i=0}^{q-k} \theta_i \theta_{i+k}, & k = 0, 1, ..., q \\ 0 & k > q \end{cases} \tag{2}$$

in which it is understood that $\theta_0 = 1$. However, if an AR(p), for given (arbitrary) order p, is specified in (1), instead of the true MA, then the vector of estimated AR coefficients is asymptotically given by $\Gamma^{-1}\gamma$, where the $p \times p$ symmetric matrix Γ has (i, j)th element γ_{i-j} and $\gamma = [\gamma_1, \, ..., \, \gamma_p]$.

The $SI(1)$ process of (1) admits the so-called vector of quarters representation (see, for example, Tiao and Grupe, 1980, Franses, 1994, or Burridge and Taylor, 2001)

$$\left(1 - L^4\right) X_\tau = U_\tau \tag{3}$$

where $X_\tau = [x_{1\tau}, x_{2\tau}, x_{3\tau}, x_{4\tau}]'$, $U_\tau = [u_{1\tau}, u_{2\tau}, u_{3\tau}, u_{4\tau}]'$. As in Burridge and Taylor (2001), it is possible to write

$$U_\tau = \sum_{j=0}^{Q} \Theta_j E_\tau \tag{4}$$

where $E_\tau = [\epsilon_{1\tau}, \epsilon_{2\tau}, \epsilon_{3\tau}, \epsilon_{4\tau}]'$ and we define the sequence of 4×4 matrices:

$$\Theta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \theta_1 & 1 & 0 & 0 \\ \theta_2 & \theta_1 & 1 & 0 \\ \theta_3 & \theta_2 & \theta_1 & 1 \end{bmatrix}, \quad \Theta_j = \begin{bmatrix} \theta_{4j} & \theta_{4j-1} & \theta_{4j-2} & \theta_{4j-3} \\ \theta_{4j+1} & \theta_{4j} & \theta_{4j-1} & \theta_{4j-2} \\ \theta_{4j+2} & \theta_{4j+1} & \theta_{4j} & \theta_{4j-1} \\ \theta_{4j+3} & \theta_{4j+2} & \theta_{4j+1} & \theta_{4j} \end{bmatrix}, \; j = 1, 2, \cdots$$

The vector MA order in (4) is $Q = [(q-1)/4] + 1$ (see Tiao and Grupe, 1980), where $[.]$ is the integer part of the expression in brackets and, for later convenience, we define $\Theta(1) \equiv \sum_{j=0}^{Q} \Theta_j$.

The lemma summarizes the stochastic characteristics of the process (1).

Lemma 1 *Consider X_τ for (3)/(4). Assuming that the elements of E_τ are independent and identically distributed with zero mean and variance σ^2, then as $N \to \infty$:*

$$\frac{1}{\sqrt{N}} X_{[rN]} \Rightarrow B(r), \quad r \in [0,1] \tag{5}$$
$$B(r) = \sigma\, \Theta(1)\, W(r)$$

where $[rN]$ denotes the integer part of rN, $B(r)$ is a 4×1 vector Brownian motion process with variance matrix $\Omega = \sigma^2 \Theta(1) \Theta(1)'$, and $W(r)$ is a 4×1 vector standard Brownian motion process with variance matrix I_4.

Here and throughout $\Rightarrow$ means convergence in distribution. This lemma can be proved along the same lines as the proof of Lemma 1 in Boswijk and Franses (1996) and del Barrio Castro (2006).

The regression for the HEGY tests, with AR augmentation of order p and no deterministic terms, takes the form

$$\begin{aligned} \Delta_4 x_{s\tau} =& \pi_0 x_{s\tau}^{(0)} + \pi_2 x_{s\tau}^{(2)} + \pi_3 x_{s\tau}^{(3)} + \pi_4 x_{s\tau}^{(4)} + \sum_{j=1}^{p} \phi_j \Delta_4 x_{s-j,\tau} + e_{s\tau}, \\ s =& 1,2,3,4, \tau = 1,2,\cdots,N \end{aligned} \tag{6}$$

The first four regressors of (6) are defined as

$$\begin{aligned} x_{s\tau}^{(0)} =& L\left(1 + L + L^2 + L^3\right) x_{s\tau} \\ x_{s\tau}^{(2)} =& -L\left(1 - L + L^2 - L^3\right) x_{s\tau} \\ x_{s\tau}^{(3)} =& -L^2\left(1 - L^2\right) x_{s\tau} \\ x_{s\tau}^{(4)} =& -L\left(1 - L^2\right) x_{s\tau} \end{aligned} \tag{7}$$

where $x_{s\tau}^{(0)}$, $x_{s\tau}^{(2)}$ are associated with the presence of the factors $(1-L)$ and $(1+L)$, respectively, of $\Delta_4 = (1-L)(1+L)\left(1+L^2\right)$, while $\left(1+L^2\right)$ is associated with the pair of variables $x_{s\tau}^{(4)}$ and $x_{s\tau}^{(3)} = x_{s-1,\tau}^{(4)}$. The overall HEGY null hypothesis of seasonal integration, $x_{s\tau} \sim SI(1)$, implies the presence of unit roots at the zero frequency (captured through π_0) and at seasonal frequencies (captured through π_2, π_3 and π_4), so that $\pi_0 = \pi_2 = \pi_3 = \pi_4 = 0$. For later convenience, we also define the autoregressive operator $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$ relating to (6).

As shown by HEGY, the regressors defined in (7) are, by construction, asymptotically orthogonal under the $SI(1)$ null hypothesis. Thus, the associated asymptotic distributions of the HEGY test statistics by considering the three factors $(1-L)$, $(1+L)$ and $\left(1+L^2\right)$ of Δ_4 one by one. Based on an ordinary least squares (OLS) estimation of this regression, t-ratio (t_{π_i}) tests are proposed to test the null of $H_0 : \pi_i = 0$, $i = 0, 2, 3, 4$, and an F-type statistic, F_{34}, is proposed to test the joint null $H_0 : \pi_3 = \pi_4 = 0$. Ghysels, Lee and Noh (1994) also propose F-type statistics to test the null hypotheses $H_0 : \pi_2 = \pi_3 = \pi_4 = 0$ and $H_0 : \pi_0 = \pi_2 = \pi_3 = \pi_4 = 0$, denoted F_{234} and F_{0234}, respectively. For further details of these tests and the asymptotic distributions for AR processes, see also Smith and Taylor (1998), Burridge and Taylor (2001) and Osborn and Rodrigues (2002).

In the next section we present the distributions of these HEGY statistics when the underlying process is (1) with nonzero MA component. Note, however, that $e_{s\tau}$ in (6) in this case cannot be serially uncorrelated. Indeed, under the true $SI(1)$ null hypothesis, $\Delta_4 x_{s\tau} = u_{s\tau}$ and hence, in the context of (6), $\phi(L)u_{s\tau} = e_{s\tau}$. Consequently, from (1), it follows that $e_{s\tau} = \phi(L)\theta(L)\epsilon_{s\tau} = \beta(L)\epsilon_{s\tau}$ and hence $e_{s\tau}$ is an MA$(p+q)$ process.

3 Asymptotics

The Proposition below presents the distributions of the t-type tests for $\pi_i = 0$ $(i = 0, 2, 3, 4)$ in (6) for a general fixed order of AR augmentation p, when the true innovations to the $SI(1)$ process are MA(q). Here we use the two functionals $A(i,j) = \int w_i(r)\, dw_j(r)$ and $D(i) = \int [w_i(r)]^2\, dr$ in which $w_i(r)$ $i = 0, 2, 3, 4$ are mutually independent standard Brownian motion processes, formed as linear combinations of the component processes of $W(r)$ in (5) of Lemma 1; see Burridge and Taylor (2001) and their Appendix for details. In an analogous way to expressions in Burridge and Taylor (2001), we further define

$$a_\phi = \sum_{k=1}^{[(p+1)/2]} (-1)^{k+1} \phi_{2k-1} = -\frac{i}{2}\left(\phi(i) - \phi(-i)\right) \tag{8}$$

$$b_\phi = 1 - \sum_{k=1}^{[p/2]} (-1)^{k} \phi_{2k} = \frac{1}{2}\left(\phi(i) + \phi(-i)\right) \tag{9}$$

where $i \equiv \sqrt{-1}$ and [.] again indicates the integer part of the extression in brackets.

The following Proposition analyzes the effect of increasing the order of AR augmentation (p) in the HEGY regression, when the DGP has the form of (1). The proof of the Proposition can be found in the Appendix.

Proposition 1 *Assume $x_{s\tau}$ follows (1) with $x_{s0} = 0$ $(s = 1, 2, 3, 4)$. Then for the t-type statistics resulting from OLS estimation of the HEGY regression (6) with given p:*

(a) the asymptotic distributions of t_{π_0} and t_{π_2} are given by:

$$t_{\pi_0} \Rightarrow \frac{\beta(1)}{\sqrt{\beta^{(2)}(1)}} \times \frac{A(0,0)}{\sqrt{D(0)}} + \frac{\sum_{k=0}^{q}\sum_{l=1+k}^{q+p}\theta_k\beta_l}{\theta(1)\sqrt{\beta^{(2)}(1)\,D(0)}} \tag{10}$$

$$t_{\pi_2} \Rightarrow \frac{\beta(-1)}{\sqrt{\beta^{(2)}(1)}} \times \frac{A(2,2)}{\sqrt{D(2)}} + \frac{\sum_{k=0}^{q}\sum_{l=1+k}^{q+p}(-1)^{k+l}\theta_k\beta_l}{\theta(-1)\sqrt{\beta^{(2)}(1)\,D(2)}}; \tag{11}$$

(b) the asymptotic distributions of t_{π_3} and t_{π_4} are given by:

$$\begin{aligned} t_{\pi_3} \;\Rightarrow\; & \frac{b_\phi\sqrt{\theta(i)\theta(-i)}}{\sqrt{\beta^{(2)}(1)}} \times \frac{[A(3,3)+A(4,4)]}{\sqrt{D(3)+D(4)}} - \frac{a_\phi\sqrt{\theta(i)\theta(-i)}}{\sqrt{\beta^{(2)}(1)}} \times \frac{[A(4,3)-A(3,4)]}{\sqrt{D(3)+D(4)}} \\ & - \frac{2\sum_{k=0}^{q}\sum_{l=2+k}^{q+p}\theta_k\beta_l\cos\left[(l-(k-2))\,\pi/2\right]}{\sqrt{\theta(i)\,\theta(-i)\,\beta^{(2)}(1)}\sqrt{D(3)+D(4)}} \end{aligned} \tag{12}$$

$$\begin{aligned} t_{\pi_4} \;\Rightarrow\; & \frac{b_\phi\sqrt{\theta(i)\theta(-i)}}{\sqrt{\beta^{(2)}(1)}} \times \frac{[A(4,3)-A(3,4)]}{\sqrt{D(3)+D(4)}} + \frac{a_\phi\sqrt{\theta(i)\theta(-i)}}{\sqrt{\beta^{(2)}(1)}} \times \frac{[A(3,3)+A(4,4)]}{\sqrt{D(3)+D(4)}} \\ & - \frac{2\sum_{k=0}^{q}\sum_{l=1+k}^{q+p}\theta_k\beta_l\sin\left[(l-k)\,\pi/2\right]}{\sqrt{\theta(i)\,\theta(-i)\,\beta^{(2)}(1)}\sqrt{D(3)+D(4)}}. \end{aligned} \tag{13}$$

In all cases, $\beta(z) = 1 + \beta_1 z + ... + \beta_{p+q}z^{p+q} = \phi(z)\theta(z)$ in which ϕ_k $(k = 1, ..., p)$ is the kth element of $\Gamma^{-1}\gamma$, where the $p \times p$ symmetric matrix Γ has (i,j)th element γ_{i-j} and the $p \times 1$ vector $\gamma = [\gamma_1, ..., \gamma_p]'$, with γ_k $(k = 1, ..., p)$ defined by (2), and $\phi_0 = 1$, while $\beta^{(2)}(z) = 1 + \beta_1^2 z + ... + \beta_{p+q}^2 z^{p+q}$.

The asymptotic distributions in Proposition 1 for a given order of augmentation p shed light on the performance of the HEGY statistics in the presence of MA innovations. In particular, (10) and (11) make clear the presence of scale and shift effects in relation to the usual Dickey-Fuller distribution, given by $A(j,j)/\sqrt{D(j)}$ $(j = 0, 2)$, that applies for these statistics applied to a seasonal random walk with $\Delta_4 x_{s\tau} = \epsilon_{s\tau}$, while (12) and (13) show such effects in relation to the distributions obtained by Burridge and Taylor (2001) for t_{π_3} and t_{π_4} with AR innovations.

Initially we focus on (10) and (11) in the following Remark.

Remark 1 *Noting that the seasonal unit root -1 has mirror image properties in relation to the zero frequency unit root +1, the scale factors $\beta(1)/\sqrt{\beta^{(2)}(1)}$, $\beta(-1)/\sqrt{\beta^{(2)}(1)}$ and the numerators of the shift terms, namely $\sum_{k=0}^{q}\sum_{l=1+k}^{q+p}\theta_k\beta_l$, $\sum_{i=0}^{q}\sum_{l=1+k}^{q+p}(-1)^{k+l}\theta_k\,\beta_l$, in (10) and (11) are equivalent to* *the corresponding terms in Proposition 1 of Galbraith and Zinde-Walsh (1999). Following the same arguments as in their Propositions 2 and 3, if p is such that $p = O(\delta \ln N)$, $\delta > 0$, then the scale factors tend to one and the numerators of the shift factors to zero. Therefore, with $p = O(\delta \ln N)$, $\delta > 0$, then*

$$t_{\pi_j} \Rightarrow \frac{A(j,j)}{\sqrt{D(j)}} \quad j = 0, 2 \tag{14}$$

which is the usual Dickey-Fuller distribution.

Now we turn attention to the t-ratio tests associated with the pair of complex conjugate unit roots at frequency $\pi/2$.

Remark 2 *The numerators of the shift factors in (12) and (13), $\sum_{k=0}^{q}\sum_{l=2+k}^{q+p}\theta_k\beta_l\cos[(l-(k-2))\pi/2]$ and $\sum_{k=0}^{q}\sum_{l=1+k}^{q+p}\theta_k\beta_l\sin[((l-k))\pi/2]$ respectively, are analogous to those of expressions (10), (11) and Proposition 1 of Galbraith and Zinde-Walsh (1999), once the HEGY transformations (7) are taken into account. Hence these terms also tend to zero if p is chosen such that $p = O(\delta \ln N)$, $\delta > 0$, then leading to the asymptotic distributions:*

$$t_{\pi_3} \Rightarrow \frac{\sqrt{\theta(i)\theta(-i)}}{\sqrt{\beta^{(2)}(1)}}\left\{\frac{b_\phi\left[A(3,3)+A(4,4)\right]-a_\phi[A(4,3)-A(3,4)]}{\sqrt{D(3)+D(4)}}\right\} \tag{15}$$

$$t_{\pi_4} \Rightarrow \frac{\sqrt{\theta(i)\theta(-i)}}{\sqrt{\beta^{(2)}(1)}}\left\{\frac{a_\phi\left[A(3,3)+A(4,4)\right]+b_\phi[A(3,4)-A(4,3)]}{\sqrt{D(3)+D(4)}}\right\}. \tag{16}$$

The distributions of (15) and (16) generalize those obtained by Burridge and Taylor (2003) for the HEGY test in the presence of AR disturbances.

Hence, it is clear that the asymptotic distributions of the t-ratio statistics associated with the unit roots at the zero and semi-annual frequencies, given by (14), remain pivotal if the order of augmentation is allowed to grow at the corresponding rate to that proposed by Galbraith and Zinde-Walsh (1999) for the non-seasonal case. But this is not the case for the t-ratio tests associated with the complex conjugate pair of roots at the annual frequency, given by (15) and (16), as previously pointed out by Burridge and Taylor (2001) for HEGY procedure in the presence of autoregressive serial correlation. These authors also find that the F-type statistics F_{34}, F_{234} and F_{0234} remain pivotal in the AR case. Therefore, the following remark examines the distribution of F-type HEGY tests in the presence of MA innovations when the order of AR augmentation is allowed to grow at the rate proposed by Galbraith and Zinde-Walsh(1999).

Remark 3 *Let the order of augmentation grow at a rate* $p = O(\delta \ln N)$, $\delta > 0$, *then using (14), (15) and (16) together with* $F_{34} \Rightarrow (1/2)\left[(t_{\pi_3})^2 + (t_{\pi_4})^2\right]$, $F_{234} \Rightarrow (1/3)\left[(t_{\pi_2})^2 + (t_{\pi_3})^2 + (t_{\pi_4})^2\right]$ *and* $F_{0234} \Rightarrow 1/4\left[(t_{\pi_0})^2 + (t_{\pi_2})^2 + (t_{\pi_3})^2 + (t_{\pi_4})^2\right]$ *due to the asymptotic orthogonality of the HEGY regressors, it is possible to write:*

$$
\begin{aligned}
F_{34} &\Rightarrow \frac{1}{2}\left(\frac{[A(3,3)+A(4,4)]^2 + [A(4,3)-A(3,4)]^2}{D(3)+D(4)}\right) \\
F_{234} &\Rightarrow \frac{1}{3}\left(\frac{A(2,2)^2}{D(2)} + \frac{[A(3,3)+A(4,4)]^2 + [A(4,3)-A(3,4)]^2}{D(3)+D(4)}\right) \qquad (17) \\
F_{0234} &\Rightarrow \frac{1}{4}\left(\frac{A(0,0)^2}{D(0)} + \frac{A(2,2)^2}{D(2)} + \right. \\
&\qquad \left. \frac{[A(3,3)+A(4,4)]^2 + [A(4,3)-A(3,4)]^2}{D(3)+D(4)}\right).
\end{aligned}
$$

These results follow since the definitions of (8) and (9) imply that $b_\phi^2 + a_\phi^2 = \left(\frac{1}{2}[\phi(i) + \phi(i)]\right)^2 + \left(-\frac{i}{2}[\phi(i) - \phi(i)]\right)^2 = \phi(i)\phi(-i)$ *and the definition of* $\beta(z)$ *implies* $\theta(i)\theta(-i)\phi(i)\phi(-i) = \beta(i)\beta(-i)$, *while, from the arguments of Galbraith and Zinde-Walsh (1999) in their Propositions 2 and 3, the scale factor* $\beta(i)\beta(-i)/\left(1 + \sum_{j=1}^{p+q}\beta_j^2\right)$ *tends to one when the order of augmentation grows at a rate* $p = O(\delta \ln N)$, $\delta > 0$.

Note that (17) implies that, as in Burridge and Taylor (2001), if the HEGY regression (6) is sufficiently augmented, all F-type tests associated with the HEGY procedure remain pivotal.

Finally, Proposition 1 also shows that the OLS estimates of the AR augmentation coefficients in (6) converge to $\Gamma^{-1}\gamma$, which are functions of the moving average coefficients in (1). Indeed, due to the asymptotic orthogonality of the stationary and nonstationary components of the DGP (1), this is the same expression as noted in Section 2 for approximating the invertible moving average component by a stationary AR process.

4 Monte Carlo Analysis

From the arguments of Galbraith and Zinde-Walsh (1999, Proposition 3), it is clear that the speed of convergence of the statistics to the asymptotic distributions derived in the previous section depend on the maximum of the moduli of the inverses of the roots of the MA polynomial $\theta(z)$. Therefore, in common with Galbraith and Zinde-Walsh (1999) and del Barrio Castro and Osborn (2008), the order of AR augmentation required to achieve limit distributions for the unit root statistics that are effectively free of nuisance parameters depends crucially on the possibility of near-cancellation of the MA roots with the AR unit roots. In the case of HEGY-type seasonal unit root tests, this possible near-cancellation applies in relation to the roots of the seasonal difference operator Δ_4. For a test statistic where near-cancellation occurs, then a substantially higher order of augmentation is required than for a case where no such cancellation applies.

This situation is illustrated in Figures 1 to 6, where we present the empirical distribution of the t-ratios $t_{\pi_1}$and t_{π_2} and the F-type statistic F_{34} in the presence of MA innovations for different orders of augmentation in (6), together with the corresponding distributions when the innovations are white noise. In addition, Table 1 presents finite sample size results for testing the unit root null hypothesis at the zero and each seasonal frequency (for quarterly data), for a nominal test size of 5%. The results in the figures and the table are based on 15,000 replications and a sample size of 400 observations, corresponding to 100 years of data (four observations per year). In all cases the data are generated from (1) with $\epsilon_{s\tau} \sim Nid(0,1)$. In Figures 1 to 4 the innovation follows a MA(1) $u_{s\tau} = \epsilon_{s\tau} + \theta_1 \epsilon_{s-1,\tau}$ process with $\theta_1 = \pm 0.8$, so that these consider cases where the MA process is close to cancellation with the unit roots ∓ 1 of Δ_4. On the other hand, the innovations used for Figures 5 and 6 follow the simple MA(2) $u_{s\tau} = \epsilon_{s\tau} + \theta_2 \epsilon_{s-2,\tau}$ process with $\theta_2 = 0.8$, corresponding to near-cancellation with the complex pair of AR unit roots implied by $1 + L^2$, and $\theta_2 = -0.8$, which is close to cancellation with $1 - L^2 = (1 - L)(1 + L)$. In order to provide further insight into the role of AR approximations for MA processes in the context of HEGY tests, the size results in Table 1 employ MA(1) processes with $\theta_1 = \pm 0.5$, ± 0.8 and simple MA(2) processes with $\theta_2 = \pm 0.5$, ± 0.8, thus providing size comparisons for a range of invertible and near-noninvertible cases. Following common empirical practice with seasonal data, we consider AR augmentation orders in (6) that are multiples of the seasonal frequency, with $p = 4, 8, 12, 16$.

Consider first the test for the zero frequency unit root using t_{π_0} in the case of positively autocorrelated MA(1) innovations. Table 1 suggests that the AR approximation works well when $\theta_1 = +0.5$ and empirical size is good for an augmentation order of 4 or above. With stronger autocorrelation ($\theta_1 = +0.8$), the test is moderately undersized when $p = 4$, but performs well for $p = 8, 12$ or 16. For this DGP, Figure 1 confirms that the augmentation order of 4 is sufficient for the empirical distribution of the test statistic to provide a reasonable approximation overall to the asymptotic distribution that applies for white noise $u_{s\tau}$ in (1), namely the Dickey-Fuller distribution (the latter is labelled df in the figure). Although the MA process has near-cancellation with the AR root of -1, this root is not under test and hence the near-cancellation is effectively irrelevant. On the other hand, when the MA coefficient is $\theta_1 = -0.8$, Figure 2 shows substantial distortion in the empirical distribution of t_{π_0} with $p = 4$ compared with the Dickey-Fuller distribution, with this distortion due to the scale and shift factors in (10). In particular, this distribution is shifted to the left (compared with the Dickey-Fuller case) and is also flatter. As evident in Table 1, this results in very substantial over-sizing of the zero frequency unit root test, with an empirical size of around 30% for a nominal size of 5%, when this order of AR augmentation is applied. However, as discussed in the previous section, for a sufficiently high order of augmentation, the distribution approaches the Dickey-Fuller one, and a reasonable approximation results in Figure 2 when $p = 12$. As anticipated, and as clearly evident in Table 1, over-sizing for this test is much less severe when $\theta_1 = -0.5$, for which $p = 4$ delivers a reasonable performance and the size is good for $p = 8$ or greater.

The situation is reversed when the HEGY statistic t_{π_2} is considered, with all size results in Table 1 for t_{π_2} and given MA coefficient θ_1 mirroring those for t_{π_0} with MA coefficient of the opposite sign. Thus, severe oversizing results for t_{π_2} in Table 1 when $\theta_1 = +0.8$ and $p = 4$, due to the scale and shift effects for this case that are clearly evident in Figure 3, and which apply because the unit root -1 is now under test. However, the empirical distribution of t_{π_2} is well approximated by the asymptotic Dickey-Fuller one in Figure 4 for the MA coefficient $\theta_1 = -0.8$, because no near-cancellation then applies at the (seasonal) frequency under test.

Corresponding results apply for the joint test F_{34} of the complex pair of roots $\pm i$ at the annual frequency. In particular, there is no relevant near-cancellation across the seasonal AR roots of $(1 + L^2)$ and the MA component $u_{s\tau} = \epsilon_{s\tau} + \theta_2 \epsilon_{s-2,\tau}$ for negative θ_2, so that size for F_{34} is generally good in these cases in Table 1, notwithstanding some over-sizing when $p = 4$. Figure 5 verifies that this applies over the whole distribution of F_{34}, with a relatively low order of augmentation being sufficient to render this empirical distribution close to the corresponding asymptotic HEGY distribution (labelled f34). However, severe size distortions apply in Table 1 for this statistic when $u_{s\tau} = \epsilon_{s\tau} + 0.8\epsilon_{s-2,\tau}$, which leads to near-cancellation with the complex pair of unit roots under test, implying that the evidence for the presence of these unit roots in the DGP is masked by the MA disturbance. Consequently, there is extreme size distortion when $p = 4$, with empirical size of 50% in Table 1. This is evident also in Figure 6, where the right-hand tail of the distribution of the test statistic is shifted upwards compared with the HEGY distribution. Although higher orders of augmentation (illustrated by $p = 8$ and 12 in Figure 6) provide better approximations to the true MA process, moving the asymptotic distribution of F_{34} towards the corresponding HEGY one, augmentation by $p = 16$ is required for good size for this case in Table 1.

The patterns shown in Table 1 and Figures 1 to 6 also carry over to cases where the joint seasonal unit root test statistics F_{234} and F_{0234} are employed. In particular, for MA processes with no root near the noninvertibility boundary, the AR approximation employed in (6) can be anticipated to perform well in practice for all HEGY test statistics. On the other hand, however, near-cancellation of the roots of the MA polynomial $\theta(z)$ with any (zero frequency or seasonal) unit root under test will lead to over-sizing when a moderate order of AR augmentation, such as $p = 4$ or 8, is used. Nevertheless, for a sufficiently high order of augmentation, tests based on the HEGY specification (6) will perform well.

5 Conclusions

This paper extends the analysis of the seasonal unit root tests of Hylleberg et al. (1990) to the case where the disturbances are generated by a moving average process. In particular, we derive the asymptotic null distributions of tests for unit roots at the zero and seasonal frequencies in a quarterly process when autoregressive augmentation is applied but the disturbances have a moving average form. We show that, provided that the regression is sufficiently augmented and that the F-type form is used for tests at the semi-annual frequency, the distributions tabulated by Hylleberg et al. (1990) continue to apply. Not surprisingly, however, if the order of augmentation is taken as some fixed value, then the distributions contain scale and shift factors in relation to the presumed distributions, with these factors being functions of the moving average coefficients and the order of augmentation applied. These results are analogous to those of Galbraith and Zinde-Walsh (1999), who study the conventional ADF test statistic in the presence of moving average disturbances, and del Barrio Castro and Osborn (2008), who derive the distributions of HEGY seasonal unit root tests when the process is periodically integrated.

This paper also establishes that, as in Burridge and Taylor (2001), the t-type statistics associated with the zero and annual frequencies together with all joint F-type statistics commonly applied to test seasonal unit roots in quarterly data, remain pivotal, provided they are sufficiently augmented. On the other hand, however, the distributions of the t-type tests associated with the pair of complex unit roots at the semi-annual frequency depend, in a non trivial manner, on nuisance parameters. These theoretical results are supported by a Monte Carlo analysis of the finite sample properties of the pivotal test statistics, which shows that high orders of AR augmentation can be required to render the empirical distributions close to the asymptotic HEGY ones when disturbances are of the MA form. Indeed, when near-cancellation applies between an MA root and a (zero or seasonal) unit root, the required order of augmentation can be very substantially greater than the default four lags often used by practitioners for quarterly data. When low augmentation orders (such as $p = 4$) are employed, the test statistics can be badly over-sized.

Although our results in (10) to (17) do not consider the inclusion of deterministic terms (such as seasonal dummies or linear trend) in the HEGY regression, our results could be easily extended to this situation. More specifically, with the inclusion of seasonal dummies or seasonal dummies and a linear trend our results will carry over when expressed using demeaned and de-trended Brownian motions. Further, as shown by Smith and Taylor (1998), the inclusion of seasonal dummies in the HEGY regression makes the test statistics invariant to starting values, while seasonal trends further provide invariance to seasonal drifts; these results apply also in our case.

References

[1] Agiakloglou, C. and Newbold, P. (1992) Empirical Evidence on Dickey-Fuller Type Tests, *Journal of Time Series Analysis*, 14, 471-483.

[2] Boswijk, H. P. and Franses P. H. (1996) Unit Roots in Periodic Autoregressions, *Journal of Time Series Analysis,* 17, 221-245.

[3] Burridge, P. and Taylor, A. M. R. (2001) On the Properties of Regression-Based Tests for Seasonal Unit Roots in the Presence of Higher-Order Serial Correlation, *Journal of Business and Economic Statistics*, 3, 374-379.

[4] del Barrio Castro, T. (2006) On the Performance of the DHF Tests against Nonstationary Alternatives, *Statistics and Probability Letters*, 76, 291-297.

[5] del Barrio Castro, T. (2007) Using the HEGY Procedure when Not All the Roots are Present, *Journal of Time Series Analysis*, 29, 910-922.

[6] del Barrio Castro, T. and Osborn, D. R. (2008) Testing for Seasonal Unit Roots in Periodic Integrated Autoregressive Processes, *Econometric Theory*, 24, 1093-1129.

[7] Franses, P. H. (1994) A Multivariate Approach to Modelling Univariate Seasonal Time Series, *Journal of Econometrics*, 63, 133-151.

[8] Fuller, W. A. (1996) *Introduction to Statistical Time Series* (2nd ed.), New York, Wiley.

[9] Galbraith, J. W. and Zinde-Walsh, V. (1999) On the Distribution of Augmented Dickey-Fuller Statistics in Processes with Moving Average Components, *Journal of Econometrics*, 93, 25-47.

[10] Ghysels, E., Lee, H. S. and Noh, J. (1994) Testing for Unit Roots in Seasonal Time Series: Some Theoretical Extensions and a Monte Carlo Investigation, *Journal of Econometrics*, 62, 415- 442.

[11] Hylleberg, S., Engle, R. F., Granger, C. W. J. and Yoo, B. S. (1990) Seasonal Integration and Cointegration, *Journal of Econometrics*, 44, 215-238.

[12] Osborn, D. R. and Rodrigues, P. M. M. (2002) Asymptotic Distributions of Seasonal Unit Root Tests: A Unifying Approach, *Econometrics Reviews*, 21,221-241.

[13] Phillips P.C.B. and Ouliaris, S. (1990) Asymptotic Tests for Residual Based Tests for Autocorrelation, *Econometrica*, 58, 165-193.

[14] Said, S.E. and Dickey, D. A. (1984) Testing for Unit Roots in Autoregressive-Moving Average Models of Unknown Order, *Biometrika*, 71, 599-607.

[15] Schwert, G. (1989) Testing for Unit Roots: A Monte Carlo Investigation, *Journal of Business and Economic Statistics*, 7, 147-158.

[16] Smith R. J. and Taylor, A. M. R. (1998) Additional Critical Values and Asymptotic Representations for Seasonal Unit Root Tests, *Journal of Econometrics*, 85, 269-288.

[17] Tiao, G.C. and Grupe, M.R. (1980) Hidden Periodic Autoregressive -Moving Average Models in Time Series Data, *Biometrika*, 67, 365-373.

6 Appendix

Proof of Proposition 1

Due to the nonstationarity of the HEGY variables $x_{s\tau}^{(0)}$, $x_{s\tau}^{(2)}$, $x_{s\tau}^{(3)}$ and $x_{s\tau}^{(4)}$ on the one hand and the stationarity of $\Delta_4 x_{s\tau}$ on the other, the coefficients associated with these two sets of regressors converge at different rates when (6) is estimated. To reflect this, define the $(4+p)\times(4+p)$ scaling matrix M as:

$$M = diag\left[4N, 4N, 4N, 4N, (4N)^{1/2}, \ldots, (4N)^{1/2}\right];$$

It is straightforward to see that the scaled OLS estimators for the HEGY regression (6) can then be expressed as

$$\begin{bmatrix} 4N\,\hat{\Pi} \\ (4N)^{1/2}\,\hat{\Phi} \end{bmatrix} = \begin{bmatrix} (4N)^{-2}\sum(Y_{s\tau}^{(NS)})(Y_{s\tau}^{(NS)})' & (4N)^{-3/2}\sum Y_{s\tau}^{(NS)}(Y_{s\tau}^{(S)})' \\ (4N)^{-3/2}\sum(Y_{s\tau}^{(S)})'Y_{s\tau}^{(NS)} & (4N)^{-1}\sum Y_{s\tau}^{(S)}(Y_{s\tau}^{(S)})' \end{bmatrix}^{-1} \times \begin{bmatrix} (4N)^{-1}\sum Y_{s\tau}^{(NS)}\Delta_4 x_{s\tau} \\ (4N)^{-1/2}\sum Y_{s\tau}^{(S)}\Delta_4 x_{s\tau} \end{bmatrix} \quad (18)$$

where

$$\hat{\Pi} = [\hat{\pi}_0, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4]';\ Y_{s\tau}^{(NS)} = [x_{s\tau}^{(0)}, x_{s\tau}^{(2)}, x_{s\tau}^{(3)}, x_{s\tau}^{(4)}]';$$

$$\hat{\Phi}^{(p)} = [\widehat{\phi}_1, \cdots, \widehat{\phi}_p];\ Y_{s\tau}^{(S)} = [\Delta_4 x_{s-1,\tau}, \cdots, \Delta_4 x_{s-p,\tau}]'$$

and the summations in (18) are over all observations, $s = 1,2,3,4$ and $\tau = 1,2,...,N$.

Note that because of the nonstationary nature of the elements of $Y_{s\tau}^{(NS)}$ and the stationarity of $Y_{s\tau}^{(S)}$ it follows that $(4N)^{-3/2}\sum Y_{s\tau}^{(NS)}(Y_{s\tau}^{(S)})' \xrightarrow{p} 0$ (where $\xrightarrow{p}$ denotes convergence in probability) and hence the inverse matrix of (18) is block diagonal. Consequently, the scaled estimators $4N\hat{\Pi}$ and $(4N)^{1/2}\,\hat{\Phi}^{(p)}$ are asymptotically orthogonal. A straightforward consequence of this block diagonality is that $\hat{\Phi} = [\sum Y_{s\tau}^{(S)}(Y_{s\tau}^{(S)})'/4N]^{-1}\sum Y_{s\tau}^{(S)}\Delta_4 x_{s\tau}/4N \rightarrow \Phi = \Gamma^{-1}\gamma$ (see also del Barrio Castro and Osborn, 2007).

Further note that $(4N)^{-2}\sum(Y_{s\tau}^{(NS)})(Y_{s\tau}^{(NS)})'$ is a 4×4 diagonal matrix due to the orthogonality of the nonstationary HEGY variables and hence their asymptotic distributions can be considered separately. To obtain the distribution of t_{π_j} in (10), (11), (12) and (13), consider first the normalized bias statistics, which can be expressed as:

$$(4N)\,\widehat{\pi}_j = \frac{(4N)^{-1}\,Y^{(j)\prime}Q\Delta_4 Y}{(4N)^{-2}\,X^{(j)\prime}QY^{(i)}} + o_p(1), \qquad j = 0,2,3,4 \quad (19)$$

where $Y^{(j)}$ and $\Delta_4 Y$ are $4N \times 1$ vectors with generic elements $x_{s\tau}^{(j)}$ $(j = 0, 2, 3, 4)$ and $\Delta_4 x_{s\tau}$ respectively, and Q is the $4N \times 4N$ matrix $Q = I - Z(Z'Z)^{-1} Z'$ where the columns of Z have generic elements $\Delta_4 x_{s-1,\tau}, \cdots, \Delta_4 x_{s-p,\tau}$.

In an analogous way to Phillips and Ouliaris (1990), it is possible to write

$$\begin{aligned}(4N)^{-2} Y^{(j)\prime} Q Y^{(j)} &= (4N)^{-2} Y^{(j)\prime} Y^{(j)} + o_p(1) \\ &= (4N)^{-1} \sum_{\tau=1}^{N} \sum_{s=1}^{4} \left(x_{s\tau}^{(j)}\right)^2 + o_p(1).\end{aligned}$$

Also,

$$\begin{aligned}(4N)^{-1} Y^{(j)\prime} Q \Delta_4 Y &= (4N)^{-1} Y^{(j)\prime} Q E^* \\ &= (4N)^{-1} Y^{(j)\prime} E^* + o_p(1)\end{aligned}$$

where E^* is a $4N \times 1$ vector with generic element $e_{s\tau}$. Due to the MA structure of $u_{s\tau}$ in the DGP (1), $e_{s\tau}$ in (6) is serially correlated, with

$$\begin{aligned}(4N)^{-1} Y^{(j)\prime} E^* &= (4N)^{-1} \sum_{s=1}^{N} \sum_{s=1}^{4} x_{s\tau}^{(j)} e_{s\tau} \\ &= (4N)^{-1} \sum_{\tau=1}^{N} \sum_{s=1}^{4} x_{s\tau}^{(j)} \left(\epsilon_{s\tau} + \beta_1 \epsilon_{s-1,\tau} + \cdots + \beta_{q+p} \epsilon_{s-(q+p),\tau}\right).\end{aligned}$$

Substituting from these expressions into (19), we obtain:

$$(4N)\,\widehat{\pi}_j = \frac{(4N)^{-1} \sum_{\tau=1}^{N} \sum_{s=1}^{4} x_{s\tau}^{(i)} \left(\epsilon_{s\tau} + \cdots + \beta_{q+p} \epsilon_{s-(q+p),\tau}\right)}{(4N)^{-2} \sum_{\tau=1}^{N} \sum_{s=1}^{4} \left(x_{s\tau}^{(i)}\right)^2} + o_p(1), \qquad j = 0, 2, 3, 4. \tag{20}$$

In order to obtain the requiered results we define the following circulant matrices $C_0 = circ\,[1, 1, 1, 1]$, $C_2 = circ\,[1, -1, 1, -1]$, $C_3 = circ\,[1, 0, -1, 0]$ and $C_4 = circ\,[0, 1, 0, -1]$ that are discused in Osborn and Rodrigues (2002). These circulant matrices C_j $(j = 0, 2, 3, 4)$ satisfy $\Theta(1)' C_0 \Theta(1) = \theta(1)^2 C_0$, $\Theta(1)' C_2 \Theta(1) = \theta(-1)^2 C_2$ and $\Theta(1)' C_3 \Theta(1) = \left(a^2 + b^2\right) C_3 = \theta(i)\,\theta(-i)\, C_3$ where $a = -\frac{i}{2}[\theta(i) - \theta(-i)]$, $b = \frac{1}{2}[\theta(i) + \theta(-i)]$ and $i = \sqrt{-1}$ (see also Osborn and Rodrigues, 2002 and del Barrio Castro and Osborn, 2008). Using (5), the denominator of (20) is given by

$$\begin{aligned}(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left(x_{s\tau}^{(j)}\right)^{2} &= (4N)^{-2}\sum_{\tau=1}^{N}4\left(X_{\tau-1}'C_{j}X_{\tau-1}\right)+o_{p}(1) \qquad j=0,2\\ &\Rightarrow \frac{\sigma^{2}}{4}\int W(r)'\,\Theta(1)'\,C_{j}\,\Theta(1)\,W(r)\,dr\\ &= \begin{cases}\sigma^{2}\theta(1)^{2}\int W^{*}(r)'\,C_{0}W^{*}(r)\,dr & j=0\\ \sigma^{2}\theta(-1)^{2}\int W^{*}(r)'\,C_{2}W^{*}(r)\,dr & j=2\end{cases} \qquad (21)\\ (4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left(x_{s\tau}^{(j)}\right)^{2} &= (4N)^{-2}\sum_{\tau=1}^{N}2\left(X_{\tau-1}'C_{3}\,X_{\tau-1}\right)+o_{p}(1) \qquad j=3,4\\ &\Rightarrow \frac{\sigma^{2}}{8}\int W(r)'\,\Theta(1)'\,C_{3}\,\Theta(1)\,W(r)\,dr\\ &= \frac{\sigma^{2}\theta(i)\,\theta(-i)}{4}\int W^{\dagger}(r)'\,C_{3}W^{\dagger}(r)\,dr. \qquad (22)\end{aligned}$$

where $W^{*}(r)=(1/2)W(r)$ and $W^{\dagger}(r)=(1/\sqrt{2})W(r)$.

Turning to the numerator of (20), it is easy to see that

$$(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(j)}\epsilon_{s\tau}=(4N)^{-1}\sum_{\tau=1}^{N}X_{\tau-1}'C_{j}E_{\tau}+o_{p}(1)\,,\quad j=0,2,3,4$$

where $E_{\tau}=[\epsilon_{1\tau},\epsilon_{2\tau},\epsilon_{3\tau},\epsilon_{4\tau}]'$. Using (5) and the identities $\Theta(1)'\,C_{0}=\theta(1)\,C_{0}$, $\Theta(1)'\,C_{2}=\theta(-1)\,C_{2}$, $\Theta(1)'\,C_{3}=bC_{3}-aC_{4}$ and $\Theta(1)'\,C_{4}=aC_{3}+bC_{4}$, it follows that

$$\begin{aligned}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(0)}\epsilon_{s\tau} &\Rightarrow \frac{\sigma^{2}}{4}\int W(r)'\,\Theta(1)'\,C_{0}dW(r)\\ &= \sigma^{2}\theta(1)\int W^{*}(r)'\,C_{0}dW^{*}(r) \qquad (23)\\ (4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(2)}\epsilon_{s\tau} &\Rightarrow \frac{\sigma^{2}}{4}\int W(r)'\,\Theta(1)'\,C_{2}dW(r)\\ &= \sigma^{2}\theta(-1)\int W^{*}(r)'\,C_{2}dW^{*}(r) \qquad (24)\\ (4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(3)}\epsilon_{s\tau} &\Rightarrow \frac{\sigma^{2}}{4}\int W(r)'\,\Theta(1)'\,C_{3}dW(r)\\ &= \frac{\sigma^{2}b}{2}\int W^{\dagger}(r)'\,C_{3}dW^{\dagger}(r)-\frac{\sigma^{2}a}{2}\int W^{\dagger}(r)'\,C_{4}dW^{\dagger}(r) \qquad (25)\\ (4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(4)}\epsilon_{s\tau} &\Rightarrow \frac{\sigma^{2}}{4}\int W(r)'\,\Theta(1)'\,C_{4}dW(r)\\ &= \frac{\sigma^{2}a}{2}\int W^{\dagger}(r)'\,C_{3}dW^{\dagger}(r)+\frac{\sigma^{2}b}{2}\int W^{\dagger}(r)'\,C_{4}dW^{\dagger}(r)\,. \qquad (26)\end{aligned}$$

Now, from the definition of the auxiliary HEGY variables $x_{s\tau}^{(0)}$ and $x_{s\tau}^{(2)}$ in (7), we have

$$
\begin{aligned}
x_{s\tau}^{(0)}\epsilon_{s-i,\tau} &= \left[x_{s-1,\tau}^{(0)} + \Delta_4 x_{s-1,\tau}\right]\epsilon_{s-i,\tau} \\
&= \left[x_{s'\tau'}^{(0)} + \sum_{k=0}^{i-1}\Delta_4 x_{s'+k,\tau'}\right]\epsilon_{s'\tau'} \quad i>0 \\
x_{s\tau}^{(2)}\epsilon_{s-i,\tau} &= \left[-x_{s-1,\tau}^{(2)} - \Delta_4 x_{s-1,\tau}\right]\epsilon_{s-i,\tau} \\
&= \left[(-1)^i x_{s'\tau'}^{(2)} + \sum_{k=0}^{i-1}(-1)^{k+i}\Delta_4 x_{s'+k,\tau'}\right]\epsilon_{s'\tau'} \quad i>0
\end{aligned}
$$

where s' and τ' indicate the quarter and year (respectively) to which $\epsilon_{s-i,\tau}$ refers. Hence, using (23) and (24), it is relatively straightforward to see that

$$
\begin{aligned}
(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(0)}\epsilon_{s-i,\tau} &= (4N)^{-1}\sum_{\tau'=1}^{N'}\sum_{s'=1}^{4}\left(x_{s'\tau'}^{(0)}\epsilon_{s'\tau'} + \sum_{k=0}^{i-1}\Delta_4 x_{s'+k,\tau'}\epsilon_{s'\tau'}\right), \qquad i=1,2,... \\
&\Rightarrow \sigma^2\theta(1)\int W^*(r)' C_0 dW^*(r) + \sigma^2\sum_{k=0}^{i-1}\theta_k \qquad (27) \\
(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(2)}\epsilon_{s-i,\tau} &= (-1)^i(4N)^{-1}\sum_{\tau'=1}^{N'}\sum_{s'=1}^{4}\left(x_{s'\tau'}^{(2)}\epsilon_{s'\tau'} + \sum_{k=0}^{i-1}(-1)^k\Delta_4 x_{s'+k,\tau'}\epsilon_{s'\tau'}\right) \\
&\Rightarrow (-1)^i\sigma^2\theta(-1)\int W^*(r)' C_2 dW^*(r) + \sigma^2\sum_{k=0}^{i-1}(-1)^{k+i}\theta_k. \qquad (28)
\end{aligned}
$$

Noting that $e_{s\tau} = \left(\epsilon_{s\tau} + \beta_1\epsilon_{s-1,\tau} + \cdots + \beta_{q+p}\epsilon_{s-(q+p),\tau}\right)$, it follows from (27) and (28) that

$$
\begin{aligned}
(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(0)}e_{s\tau} &\Rightarrow \sigma^2\theta(1)\beta(1)\int W^*(r)' C_0 dW^*(r) \\
&\quad +\sigma^2\sum_{k=0}^{q}\sum_{l=k+1}^{p+q}\theta_k\beta_l \qquad (29) \\
(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(2)}e_{s\tau} &\Rightarrow \sigma^2\theta(-1)\beta(-1)\int W^*(r)' C_2 dW^*(r) \\
&\quad +\sigma^2\sum_{k=0}^{q}\sum_{l=k+1}^{p+q}(-1)^k\theta_k(-1)^l\beta_l. \qquad (30)
\end{aligned}
$$

Finally, using (21), (29) and (30), the distributions of the normalized bias statistics $(4N)\widehat{\pi}_0$ and $(4N)\widehat{\pi}_2$ are easily obtained as

$$
(4N)\widehat{\pi}_0 \Rightarrow \frac{\beta(1)\theta(1)\int W^*(r)' C_0 dW^*(r) + \sum_{k=0}^{q}\sum_{l=k+1}^{p+q}\theta_k\beta_l}{\theta(1)^2\int W^*(r)' C_0 W^*(r)\,dr} \qquad (31)
$$

$$
\begin{aligned}
(4N)\widehat{\pi}_2 \Rightarrow\ & \frac{\beta(-1)\int W^*(r)' C_2 dW^*(r)}{\theta(-1)\int W^*(r)' C_2 W^*(r)\,dr} \\
& + \frac{\sum_{k=0}^{q}\sum_{l=k+1}^{p+q}(-1)^k\theta_k(-1)^l\beta_l}{\theta(-1)^2\int W^*(r)' C_2 W^*(r)\,dr}. \qquad (32)
\end{aligned}
$$

Now for $(4N)\,\widehat{\pi}_3$ and $(4N)\,\widehat{\pi}_4$, note first that the relationship between $x_{s\tau}^{(3)}$ and $x_{s\tau}^{(4)}$, implies that $x_{s\tau}^{(3)} = x_{s-1,\tau}^{(4)} = -x_{s-2,\tau}^{(3)} - \Delta_4 x_{s-2,\tau}$. Hence using the definition of the HEGY auxiliary variables $x_{s\tau}^{(3)}$ and $x_{s\tau}^{(4)}$ in (7), it is possible to write:

$$\begin{aligned}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\epsilon_{s-i,\tau} &= (-1)^{\frac{i}{2}}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\Bigg(x_{s',\tau}^{(3)}\epsilon_{s'\tau'} \\ &\quad + \sum_{k=0}^{(i/2)-1}(-1)^k \Delta_4 x_{s'+2k,\tau'}\epsilon_{s'\tau'}\Bigg) \qquad i \;\; even \\ &\Rightarrow (-1)^{\frac{i}{2}}\frac{\sigma^2}{2}\left(b\int W^{\dagger}(r)' C_3 dW^{\dagger}(r) - a\int W^{\dagger}(r)' C_4 dW^{\dagger}(r)\right) \\ &\quad + \sum_{k=0}^{(i/2)-1}(-1)^{k+\left(\frac{i}{2}\right)}\theta_k\sigma^2 \end{aligned} \tag{33}$$

$$\begin{aligned}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\epsilon_{s-i,\tau} &= (-1)^{\frac{(i-1)}{2}}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\Bigg(x_{s',\tau'}^{(3)}\epsilon_{s'+1,\tau'} + \\ &\quad + \sum_{k=0}^{[(i-1)/2]-1}(-1)^k \Delta_4 x_{s'+2k,\tau'}\epsilon_{s'+1,\tau'}\Bigg) \qquad i \;\; odd \\ &\Rightarrow (-1)^{\frac{(i-1)}{2}}\frac{\sigma^2}{2}\left(a\int W^{\dagger}(r)' C_3 dW^{\dagger}(r) + b\int W^{\dagger}(r)' C_4 dW^{\dagger}(r)\right) \\ &\quad + \sum_{k=0}^{[(i-1)/2]-1}(-1)^{k+(i-1)/2}\theta_{k+1}\sigma^2 \end{aligned} \tag{34}$$

Using (25), (26), (33), (34) and $x_{s\tau}^{(3)} = x_{s-1,\tau}^{(4)} = -x_{s-2,\tau}^{(3)} - \Delta_4 x_{s-2,\tau}$, the numerators of $(4N)\,\widehat{\pi}_3$ and $(4N)\,\widehat{\pi}_4$ become

$$\begin{aligned}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)} e_{s\tau} \;\Rightarrow\; & \frac{\sigma^2\theta(i)\,\theta(-i)}{2}\Bigg\{\left(1-\sum_{k=1}^{[p/2]}(-1)^k\phi_{2k}\right)\int W^{\dagger}(r)' C_3 dW^{\dagger}(r) \\ & -\left(\sum_{k=1}^{[(p+1)/2]}(-1)^{k+1}\phi_{2k-1}\right)\int W^{\dagger}(r)' C_4 dW^{\dagger}(r)\Bigg\} \\ & -\sigma^2\sum_{k=0}^{q}\sum_{l=k+2}^{q+p}\theta_k\cos\left[(l-(k-2))\,\pi/2\right]\beta_l \end{aligned} \tag{35}$$

$$\begin{aligned}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(4)} e_{s\tau} \;\Rightarrow\; & \frac{\sigma^2\theta(i)\,\theta(-i)}{2}\Bigg\{\left(\sum_{k=1}^{[(p+1)/2]}(-1)^{k+1}\phi_{2k-1}\right)\int W^{\dagger}(r)' C_3 dW^{\dagger}(r) \\ & +\left(1-\sum_{k=1}^{[p/2]}(-1)^k\phi_{2k}\right)\int W^{\dagger}(r)' C_4 dW^{\dagger}(r)\Bigg\} \\ & -\sigma^2\sum_{k=0}^{q}\sum_{l=k+1}^{q+p}\theta_k\sin\left[((l-i))\,\pi/2\right]\beta_{l.} \end{aligned} \tag{36}$$

where we have used the identities

$$\begin{aligned}
\theta(i)\,\theta(-i) &= a^2+b^2 \\
\left(a^2+b^2\right)\left(1-\sum_{k=1}^{[p/2]}(-1)^k\phi_{2k}\right) &= \left(b+\beta_1 a-\beta_2 b-\beta_3 a+\beta_4 b+\beta_5 a-\beta_6 b-\beta_7 a+\cdots\right) \\
\left(a^2+b^2\right)\left(\sum_{k=1}^{[(p+1)/2]}(-1)^{k+1}\phi_{2k-1}\right) &= \left(a-\beta_1 b-\beta_2 a+\beta_3 b+\beta_4 a-\beta_5 b-\beta_6 a+\beta_7 b+\cdots\right).
\end{aligned}$$

Hence from (35), (36) and (22) it is possible to obtain the asymptotic distributions of $(4N)\,\widehat{\pi}_3$ and $(4N)\,\widehat{\pi}_4$ as

$$\begin{aligned}
(4N)\,\widehat{\pi}_3 \Rightarrow\; & \frac{\left(1-\sum_{k=1}^{[p/2]}(-1)^k\phi_{2k}\right)\int W^{\dagger}(r)'\,C_3 dW^{\dagger}(r)}{\frac{1}{2}\int W^{\dagger}(r)'\,C_3 W^{\dagger}(r)\,dr} \\
& -\frac{\left(\sum_{k=1}^{[(p+1)/2]}(-1)^{k+1}\phi_{2k-1}\right)\int W^{\dagger}(r)'\,C_4 dW^{\dagger}(r)}{\frac{1}{2}\int W^{\dagger}(r)'\,C_3 W^{\dagger}(r)\,dr} \\
& -\frac{\sum_{k=0}^{q}\theta_k\sum_{l=k+2}^{q+p}\cos\left[(l-(k-2))\,\pi/2\right]}{\frac{\theta(i)\theta(-i)}{4}\int W^{\dagger}(r)'\,C_3 W^{\dagger}(r)\,dr}
\end{aligned} \tag{37}$$

$$\begin{aligned}
(4N)\,\widehat{\pi}_4 \Rightarrow\; & \frac{\left(\sum_{k=1}^{[(p+1)/2]}(-1)^{k+1}\phi_{2k-1}\right)\int W^{\dagger}(r)'\,C_3 dW^{\dagger}(r)}{\frac{1}{2}\int W^{\dagger}(r)'\,C_3 W^{\dagger}(r)\,dr.} \\
& +\frac{\left(1-\sum_{k=1}^{[p/2]}(-1)^k\phi_{2k}\right)\int W^{\dagger}(r)'\,C_4 dW^{\dagger}(r)}{\frac{1}{2}\int W^{\dagger}(r)'\,C_3 W^{\dagger}(r)\,dr.} \\
& -\frac{\sigma^2\sum_{k=0}^{q}\theta_k\sum_{l=k+1}^{q+p}\sin\left[((l-k))\,\pi/2\right]}{\frac{\theta(i)\theta(-i)}{4}\int W^{\dagger}(r)'\,C_3 W^{\dagger}(r)\,dr}.
\end{aligned} \tag{38}$$

Finally note that from (31), (32), (37) and (38), it is possible to establish that $(4N)\,\widehat{\pi}_j = O_p(1)$ for $j = 0$, 2, 3 and 4, hence $\widehat{\pi}_j = o_p(1)$. Since, from (6), $e_{s\tau} = \Delta_4 x_{s\tau} - \sum_{j=1}^{p}\phi_j \Delta x_{s-j,\tau} = \beta(L)\,\epsilon_{s\tau}$ under the $SI(1)$ null hypothesis, then $\hat{\sigma}^2 \to \sigma^2\left(1+\sum_{j=1}^{p+q}\beta_j^2\right)$. Using

$$t_{\pi_j} = \hat{\sigma}^{-1}(4N)\,\widehat{\pi}_i \times \sqrt{(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left(x_{s\tau}^{(j)}\right)^2} + o_p(1) \qquad j=0,2,3,4 \tag{39}$$

these results in conjunction with (31), (32), (37), (38), (21) and (22) yield the asymptotic distributions of Proposition 1. In particular, note that the ranks of C_0, C_2 and C_3, C_4 are one and two respectively, and we can write:

$$
\begin{aligned}
C_0 &= v_0 v_0' \qquad C_2 = v_2 v_2' \qquad C_3 = v_3 v_3' \qquad C_4 = v_3 v_4' \\
v_0' &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\
v_2' &= \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix} \\
v_3' &= \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \\
v_4' &= \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
w_0(r) &= v_0' W^*(r) = \frac{1}{2} v_0' W(r) = \frac{1}{\sqrt{4}} \sum_{i=1}^{4} W_i(r) \qquad (40) \\
w_2(r) &= v_2' W(r) = \frac{1}{2} v_2' W(r) = \frac{1}{\sqrt{4}} \sum_{i=1}^{4} (-1)^i W_i(r) \\
v_3' W^\dagger(r) &= \frac{1}{\sqrt{2}} \begin{bmatrix} W_4(r) - W_2(r) \\ W_1(r) - W_3(r) \end{bmatrix} = \begin{bmatrix} w_3(r) \\ w_4(r) \end{bmatrix} \\
v_4' W^\dagger(r) &= \frac{1}{\sqrt{2}} \begin{bmatrix} W_3(r) - W_1(r) \\ W_4(r) - W_2(r) \end{bmatrix} = \begin{bmatrix} -w_4(r) \\ w_3(r) \end{bmatrix}.
\end{aligned}
$$

With $W_i(r)$ as the ith element of the 4×1 vector Brownian motion $W(r)$, (40) defines $w_i(r)$ $i = 0, 2, 3, 4$ as the mutually independent univariate standard Brownian motions of Proposition 1, obtained as mutually orthogonal transformations of $W(r)$. The results in Proposition 1 then follow, by using the functionals $A(i,j) = \int w_i(r) dw_j(r)$ $(i, j = 0, 2, 3, 4)$ and $D(i) = \int [w_i(r)]^2 dr$ $(i = 0, 2, 3, 4)$, together with the definitions of (8) and (9).

Table 1. Empirical size of HEGY unit root test statistics.

	DGP	HEGY Test AR Augmentation (p)			
Statistic	MA polynomial	4	8	12	16
t_0	$1+0.5L$	0.0451	0.0478	0.0474	0.0463
	$1+0.8L$	0.0349	0.0431	0.0438	0.0437
	$1-0.5L$	0.0613	0.0485	0.0484	0.0467
	$1-0.8L$	0.3009	0.1164	0.0686	0.0533
t_2	$1+0.5L$	0.0605	0.0475	0.0445	0.0439
	$1+0.8L$	0.3063	0.1172	0.0705	0.0541
	$1-0.5L$	0.0460	0.0470	0.0481	0.0465
	$1-0.8L$	0.0329	0.0411	0.0466	0.0463
F_{34}	$1+0.5L^2$	0.0645	0.0490	0.0471	0.0489
	$1+0.8L^2$	0.5006	0.2022	0.0952	0.0561
	$1-0.5L^2$	0.0559	0.0508	0.0508	0.0495
	$1-0.8L^2$	0.0644	0.0555	0.0507	0.0472

Notes: values shown are the empirical sizes for the HEGY unit root test statistics for AR augmentation orders $p = 4, 8, 12, 16$ in (6) when the true process is given by (1) with MA polynomial $\theta(L)$ as indicated. All quoted results are based on 15,000 replications for a nominal test size of 0.05, with $S = 4, N = 100$, and hence total observations $T = 400$. The critical values used in these calulations are -1.934 for t_0 and t_2, and 3.106 for F_{34}, which have been obtained for $T = 400$ using white noise $u_{s\tau}$ in (1) and 100,000 replications.

Figure 1: Empirical distribution of t_{π_0} for $\theta(L) = 1 + .8L$ when $p = 4$ and 8.

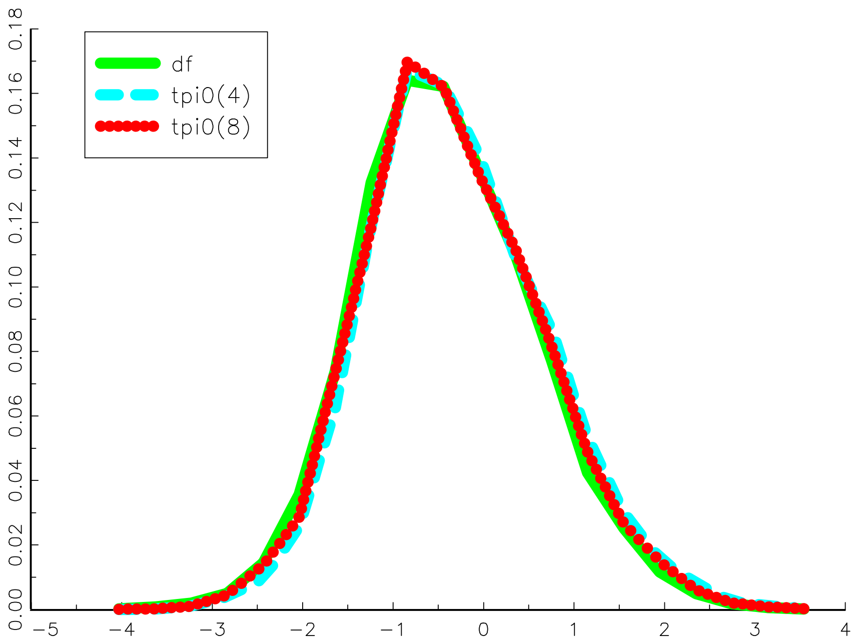

Figure 2: Empirical distribution of t_{π_0} for $\theta(L) = 1 - .8L$ when $p = 4$, 8 and 12.

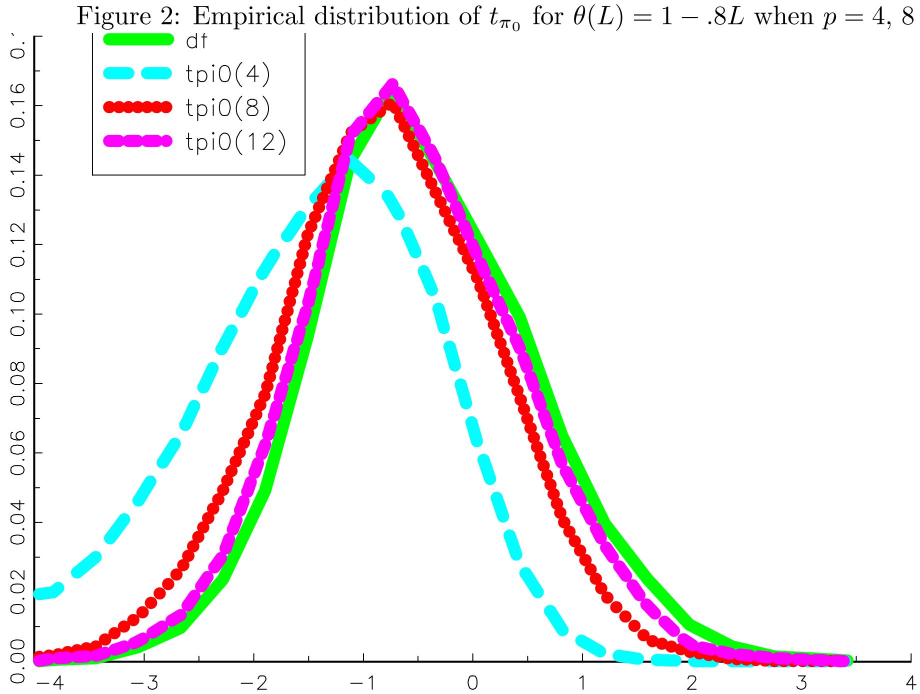

Figure 3: Empirical distribution of t_{π_2} for $\theta(L) = 1 + .8L$ when $p = 4, 8$ and 12.

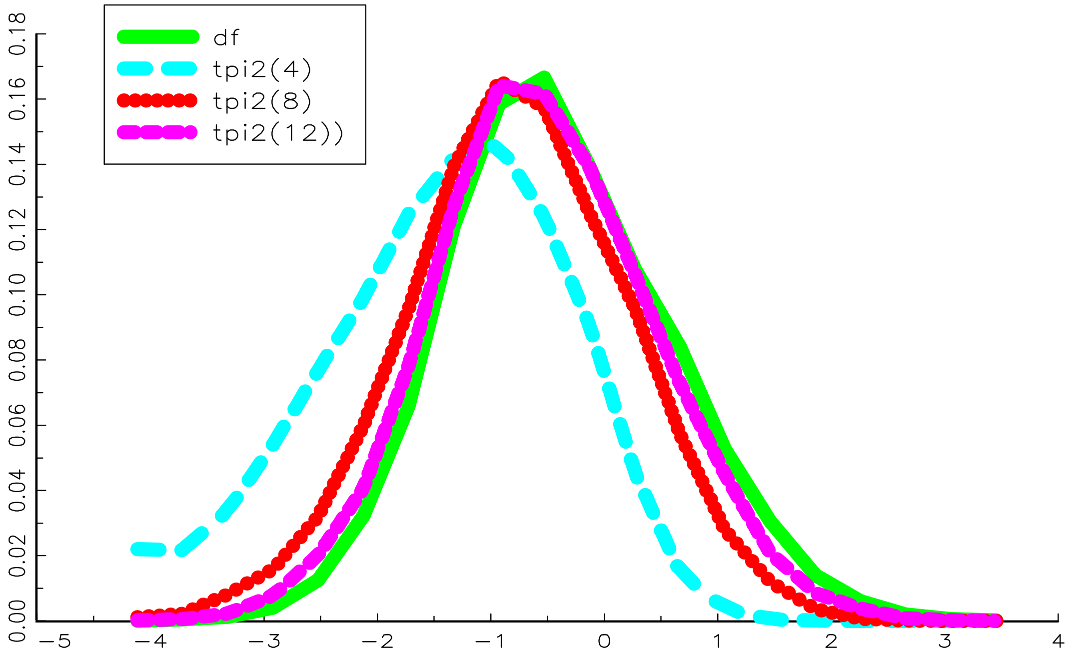

Figure 4: Empirical distribution of t_{π_2} for $\theta(L) = 1 - .8L$ when $p = 4$ and 8.

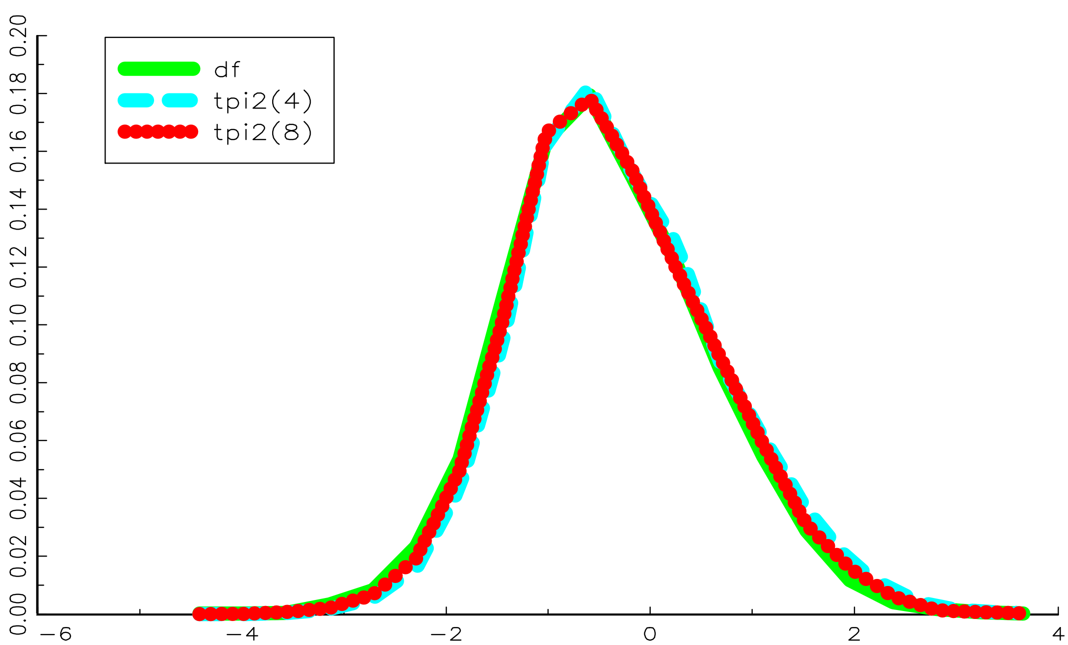

Figure 5: Empirical distribution of F_{34} for $\theta(L) = 1 - .8L^2$ when $p = 4$ and 8.

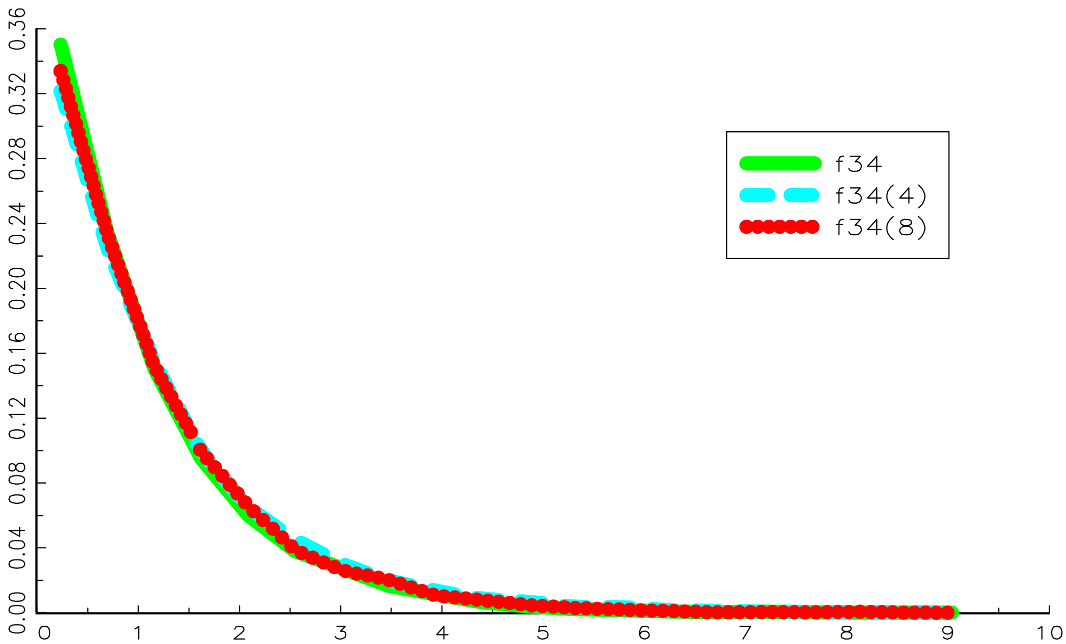

Figure 6: Empirical distribution of F_{34} for $\theta(L) = 1 + .8L^2$ when $p = 4$, 8 and 12.