Scientific collaboration networks: how little differences can matter a lot*

Antoni Rubí-Barceló†

November, 2008

Abstract

Empirical studies such as Goyal, van der Leij and Moraga (2006) or Newman (2004) show that scientific collaboration networks present a highly unequal and hierarchical distribution of links. This implies that some researchers can be much more active and productive than others and, consequently, they can enjoy a much better scientific reputation. One may think that big intrinsical differences among researchers can constitute the main driving force behind these huge inequalities. We propose a model that show how almost identical individuals self-organize themselves in a very unequal and hierarchical structure as is observed in the real-world co-authorship networks. In consequence, this model provides an incentives-based explanation of that empirical evidence.

JEL classification: Z13, D85

Keywords: network formation game, scientific collaboration.

*Previous drafts of this paper were circulated and presented under the title "Scientific collaboration networks: the role of Heterogeneity and Congestion". I am grateful to Antonio Cabrales and Marco van der Leij for very helpful comments and suggestions. I have also benefitted from comments by Joan de Martí, David Grandadam, Coralio Ballester, Alfonso Rosa-García, Adrien Vigier, Fernando Vega-Redondo, Fabrizio Germano, Tomás Lejarraga and one anonymous referee. Thanks also to other participants in *Augustin Cournot Doctoral Days 2007*, *Economics and Management of Networks 2007*, *Conference of the Association of Southern European Economic Theorists 2007*, *Simposio de Análisis Económico 2007*, and *Theory Workshop* at Universitat d'Alacant. Financial support from Universitat Pompeu Fabra and Fundación Ramón Areces is gratefully acknowledged. All errors are of course mine.

†Universitat de les Illes Balears, Cra. Valldemossa Km. 7.5. 07122 Palma de Mallorca. e-mail: antoni.rubi@uib.es

1 Introduction

Social networks underlie many economic and social activities to the point that certain outcomes cannot be understood without taking into account the specific network structure. Examples and references are numerous[1]. One of the environments in which the key role of a social network is more evident is academics. In scientific production, the association with a group of competent colleagues to exchange information is a strong advantage in order to discover errors, raise research questions, and discern the appropriate ways to solve a problem. This unquestionable significance of networks in understanding scientific activity is one of the reasons that explain the extensive empirical work on this field. Today, in the advent of the information and communication revolution, data on scientific articles and researchers is stored in electronic databases containing thousands of records. With the use of these databases, empirical studies are able to reproduce co-authorship networks (in these networks a link between two researchers exists whenever there exists an article coauthored by them). From there, they are able to represent and analyze the main statistics of the collaboration among researchers.

Empirical research about co-authorship networks is large[2]. Newman (2004), Newman (2001a) and Newman (2001b) analyze the defining statistics of co-authorship networks in Biology, Physics and Mathematics. Laband and Tollison (2000) focus on the importance of informal collaboration relationships in the comparison between networks in Economics and Biology. Hudson (1996) studies the reasons of the increase in the number of coauthors per paper in Economics. But the empirical work that most clearly shows these patterns of collaboration is Goyal, van der Leij and Moraga (2006) (GVM hereafter). This work describes a detailed image of the features of actual co-authorship networks[3].

In spite of the great variety of empirical studies, there is a lack of foundational theoretical models that analyze how individual decisions contribute to the formation of scientific collaboration networks. To the best of our knowledge, chapter 4 in van der Leij (2006) is the only attempt to compensate this deficiency. This paper, proposes a model that differs from van der Leij (2006) but shares the same objective.

[1] Calvó-Armengol and Jackson (2004) on learning about job openings through contacts or Kranton and Minehart (2001) on buyer-seller networks are only two examples.

[2] Albert and Barabási (2002) offers a survey of empirical studies about any type of networks.

[3] Although this empirical work refers to the field of Economics, we will argue that the main characteristics of co-authorship networks apply to other fields.

1.1 Characteristics of co-authorship networks

Before introducing the model, let us describe some of the key features of scientific collaboration networks. A surprising characteristic is the small average distance (measured by the shortest path length) between pairs of nodes. This stylized fact of social networks is captured in the famous "six degrees of separation" of John Guare's play[4]. Scientific collaboration networks are not an exception to this phenomenon as GVM shows. The average distance in the Economics co-authorship network they analyzed was 9.47 with a total population of 33,027 nodes (*i.e.* researchers). This regularity extends to other fields. Newman (2004) shows that the average distances are 4.6 in Biology, 5.9 in Physics and 7.6 in Mathematics.

The main features we will focus on refer to the degree-distribution of nodes which tends to show that a small part of the population accumulates a large proportion of links, *i.e.* there is a strong inequality among agents. In particular, GVM found that the 20% of most-linked authors in Economics account for about 60% of all the links. Newman (2004) shows that this phenomenon also extends to co-authorship networks in the fields of Biology, Physics and Mathematics. In each case, the distribution is fat tailed, with a small fraction of scientists having a very large number of collaborators. Moreover, network structures are hierarchical. GVM shows that the best-connected researchers collaborate extensively and most of their coauthors do not collaborate with each other. On the other hand, Newman (2004) found that most of the connections (64%) of an individual's shortest path to other researchers pass through the best-connected of their collaborators, and most of the remainders pass through the next-best connected. GVM illustrates these findings through the local network of J. Stiglitz represented in Figure 1.

These results lead GVM to conclude that: "the world of Economics is spanned by inter-linked stars" (an inter-linked star is a network in which some nodes connected among them accumulate a lot of links with other nodes who are not connected among themselves). Despite that there is no such conclusion referred to co-authorship networks in other fields, the similarity in the general results showed in Newman (2004) suggests a similar pattern in Biology, Physics and Mathematics. Moreover, GVM analyzes the evolution over the last thirty years and concludes that such a structure is stable over time.

[4]Stanley Milgram (1967) pioneered the study of path length through a clever experiment where people had to send a letter to another person who was not directly known to them. In the literature, the diameters of a variety of networks have been measured. These include purely social networks, co-authorship networks, parts of the internet and parts of the world wide web. See Albert and Barabási (2002) for an illuminating account.

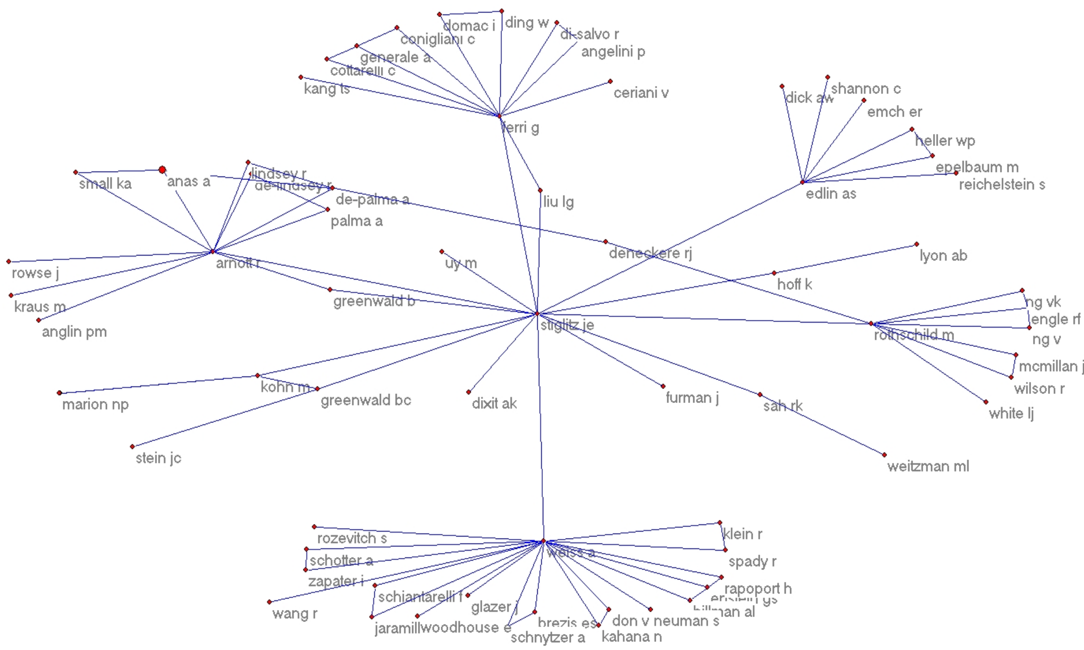

Figure 1: Local network of J. Stiglitz in 1990's

1.2 Preview of the model and results

This paper shows that the effects of some simple driving forces can explain the formation of unequal and hierarchical scientific collaboration networks as is observed in the real-world. Moreover, we do not need to assume huge *a priori* differences among researchers to reproduce this kind of structures. In our model, these forces are both the scarcity of original ideas and the benefits from cooperation. These forces stimulate scientific collaboration and they are caused by the heterogeneity among researchers and by their limited processing and creative capacities. Specifically, agents in our model are heterogeneous in terms of their level of talent (which directly affects the value of their scientific contributions); on the other hand, their limited processing capacity imposes an upper bound in the number of scientific contributions they can produce and their limited creative capacity fixes an upper bound in the number of ideas they can create. We propose a simple static model of network formation in which individuals make decisions concerning the intensity of their collaboration relationships with other researchers through a link formation game. The decision of whether to form a collaboration link must consider the trade off between the rewards from collaborating with more productive researchers and the costs derived from using part of their limited processing capacity.

After introducing the defining elements of our model, the basic assumptions, the payoff function, and the equilibrium concept in section 2, we characterize the equilibrium networks in section 3. In particular, the results show that in equilibrium, for any allocation of talents and for a broad family of production functions, some agents will be able to collaborate with many others and exhaust, is such a way, their processing capacity. Contrarily, the rest of researchers will have a lack of collaborators and consequently they will not receive a sufficient amount of ideas to exhaust their processing capacity; for this reason, they will have a much lower scientific productivity. In consequence we show that, regardless of how small are the *a priori* differences among researchers, equilibrium networks can be very unequal in terms of agents' productivity.

Our model reproduces a natural network formation process that allows us to figure out the conditions under which self-interested researchers will organize themselves forming the scientific collaboration networks observed in reality. In consequence, the model provides an incentives-based explanation of the actual shape of this kind of networks.

1.3 Literature Review

Theoretical models of social network formation can be classified into two groups. On one hand, there is the physics-based modeling of society. This approach treats agents as if they were just matter. That is, agents are non-strategic. This set has its origins in the random graph literature and has examples in sociology and recently in computer science and statistical physics. References of this kind of models are abundant[5] but we will focus on two of them. Jackson and Rogers (2006) proposes a nice, simple and general model of network formation. The authors combine random meeting and network-based meeting in a natural manner and analyze the relevance of these two forces in determining the formation of different kinds of networks (scientific collaboration structures are one of them). The second model we focus on is Arenas et al (2003). The authors present a stylized model of a problem-solving organization –whose internal communication structure is given by a network– that can suffer congestion. The authors develop a design problem to determine which kind of network architectures optimizes performance for any given problem arrival rate. Contrarily to our model, the network is fixed and players are non strategic.

The second classification of models involves strategic formation of networks and use game theoretic tools. That is, there is no exogenous prescription of how the network is formed but there is a definition

[5]See Newman (2003) for a survey. Some examples are Watts (1999), Cooper and Frieze (2003) or Price (1976).

of the rules of the game that agents have to play to form the network (see Jackson (2004) for a survey of this type of models). The model presented here belongs to this group of models. As introduced above, the work that more closely relates to our model is chapter 4 in Van der Leij (2006). This author also attempts to develop a theoretical model to explain the empirical regularities of research collaboration networks. In both models, heterogeneity across researchers plays a key role in explaining the results. Contrarily to our paper, Van der Leij constructs a model in which the cost of link formation and the specific academic rewards scheme affect the equilibrium network topologies. Our model involves the limited processing capacity and heterogeneity across agents as the key factors for obtaining the results. Moreover, we do not require a minimum degree of heterogeneity among researchers (as Van der Leij (2006) does) to reproduce the huge inequalities observed in reality.

2 General setting

Let N be the set of agents, interpreted as researchers, and let $n = |N|$ be a sufficiently large number. Each researcher is characterized by her level of talent which is exogenous, fixed, and has been randomly generated from a continuous distribution function[6]. In consequence, researchers can be ordered by their level of talent in a well defined ranking. We use natural numbers to label agents according to their position in that ranking. Thus, agent i has $i-1$ researchers with a higher talent. Let h be the vector of talent endowments and h_i be the i-th element of this vector interpreted as researcher i's amount of talent. Notice that $h_i > h_j$ for any pair of agents such that $j > i$. The object of the researchers of this model is to maximize the number and value of their scientific contributions. A researcher can participate in a new contribution either as creator of the original idea or as processor. Each agent can play both roles.

- Researchers as creators. All agents have a creative capacity that allows them to generate ρ original ideas.

- Researchers as processors. These original ideas need to be processed to become a scientific contribution. This processing work can be done either by the original creator of the idea or by some collaborator. We assume that all researchers have a limited processing capacity that we normalize to one.

[6] The probability of having two agents with the same amount of talent is zero.

Notice that all researchers of this model are exactly identical with respect to these two faculties, *i.e.* all of them have exactly the same creative and processing capacities. Heterogeneity among agents arises with respect to the value of a particular contribution. We assume that this value depends on the talent of both the creator and the processor (notice that, for a particular contribution, these two roles can be played by the same agent). Specifically, $f(h_i, h_j)$ denotes the value of a contribution in which i creates the original idea and j processes it (or viceversa). We assume a positive relationship between the talents of both creator and processor and the value of their scientific contribution.

Assumption 1 *For any* $i, j, k \epsilon N$ *such that* $i \neq j$, $f(h_i, h_j) > f(h_i, h_k)$ *whenever* $h_j > h_k$.

In words, for any researcher the higher is the talent of a collaborator the higher is the value of a shared scientific contribution. Additionally, for $k = i$ this assumption implies that for any researcher $i \epsilon N$ the value of a contribution shared with a more talented researcher is higher than the value of a single-authored contribution[7]. Another important assumption of the model is related to the size of ρ, the creative capacity of agents.

Assumption 2 $\rho < 1$.

This assumption implies that agents are able to process more ideas than what they are able to create by their own means. In consequence, researchers need original ideas from others to exhaust their processing capacity (equal to 1). This scarcity of original ideas pushes the agents of our model to accept ideas from others. On the other hand, assumption 1 implies that agents have incentives to send their original ideas to others in order to be processed. These flows of ideas are interpreted here as scientific collaboration. Therefore, both the scarcity of original ideas and the benefits from cooperation stimulate scientific collaboration in our model.

2.1 Description of the game

The agents' strategic variables refer to the election of collaborators. Specifically, each agent i will choose the n-dimensional vectors q_i and p_i. The vector $q_i = (q_{i1}, q_{i2}, ..., q_{in})$ refers to the role of agent i as creator. In particular, $q_{ij} \epsilon [0, 1]$ denotes the proportion of i's original ideas sent to j to process. In consequence, $\sum_{j \epsilon N} q_{ij} = 1$, $\forall i \epsilon N$. Thus, $q_{ij} = 0$ means that i will not send any idea to j, therefore i does not consider j as a potential processor of her ideas. A positive q_{ij} implies that i will transmit some

[7]Notice that we do not impose any restriction on the comparison between the value of a single-authored contribution by agent i and the value of a contribution in which i collaborates with a less talented researcher.

original ideas to j. Let $N_i = \{j \in N : q_{ji} > 0\}$ be the set of players who send original ideas to i and $M_i = \{j \in N : q_{ij} > 0\}$ be the set of destinations of i's original ideas. The vector $p_i = (p_{1i}, p_{2i}, ..., p_{ni})$ refers to the role of agent i as processor. In particular, $p_{ji} \epsilon [0, 1]$ denotes the processing effort that agent i invests on each of the ideas coming from agent j. The above mentioned processing capacity constraint implies that $\rho \sum_{j \epsilon N} q_{ji} p_{ji} \leq 1$ for all $i \in N$. So, $p_{ji} = 0$ means that researcher i invests no time to process ideas coming from j, therefore i does not consider j as a potential source of original ideas. Whenever $p_{ji} > 0$ agent i will invest some processing effort to the ideas coming from j.

Agents i and j are collaborators if either $p_{ij} q_{ij} > 0$ (i is the creator and j is the processor) or $p_{ji} q_{ji} > 0$ (j is the creator and i is the processor). In any case, mutual consent is required to establish a scientific collaboration relationship. Let Q and P be the $n * n$ matrices agglomerating the vectors q_i and p_i for all $i \epsilon N$. Notice that these scientific collaboration relationships can be represented by weighted and directed links. In particular, there exists a link from i to j if and only if $p_{ij} q_{ij} > 0$. Moreover, $p_{ij} q_{ij} \epsilon [0, 1]$ can be interpreted as the collaboration intensity in the flow of ideas from agent i (creator) to agent j (processor). According to this interpretation, the $n * n$ matrix $G \equiv Q \otimes P$ is a directed graph on N, where entry g_{ij} denotes the intensity of the scientific collaboration relationship in which i creates original ideas and j processes them.

Agents play a one-shot game that can be structured in three stages:

Stage 1. Link formation game: all players strategically and simultaneously announce their q and p vectors. Formally, the strategy space for player i is $S_i = [0, 1]^{2n}$. A particular strategy s_i is a pair (q_i, p_i). A strategy profile $s = (s_1, ..., s_n)$ induces a directed-weighted graph $G(s) = Q \otimes P$. We shall use the pair (q_i, p_i) to denote agent i's strategy.

Stage 2. Creation and distribution of ideas: once the scientific collaboration network is formed, each researcher $i \epsilon N$ creates ρ ideas and distributes them according to q_i.

Stage 3. Processing of ideas and resolution: Each researcher $i \epsilon N$ process ideas according to p_i. Then, participants in all scientific contributions receive their payoff.

As commented above, the object of the researchers of this model is to maximize the number and value of their scientific contributions. Specifically, the agent i's payoff can be written as:

$$\Pi_i(Q, P) = \rho[\sum_{j \in N} q_{ji} p_{ji} f(h_j, h_i) + \sum_{l \in N \setminus i} q_{il} p_{il} f(h_i, h_l)] \tag{*}$$

The first part of this function represents the payoff derived from the contributions where i acts as a processor whereas the second part represents the payoff obtained from the contributions in which i is the creator. These payoffs are the result of multiplying the value of the particular contribution (which depends on the collaborators' talents) by the intensity of their collaboration relationship. Payoff can be interpreted as the value of the expected number of contributions.

Each researcher $i\epsilon N$ has to choose the optimal pair (q_i, p_i) satisfying the following restrictions:

$$\rho \sum_{j\epsilon N} q_{ji} p_{ji} \leq 1 \tag{1}$$

$$0 \leq q_{il} \leq 1 \ \forall \ l\epsilon N \text{ and } \sum_{l\in N} q_{il} = 1 \tag{2}$$

$$0 \leq p_{ji} \leq 1 \ \forall \ j\epsilon N \tag{3}$$

The first restriction represents the limited processing capacity of agents. Restrictions (2) and (3) derive from the definition of q_i and p_i, respectively. For a given pair (q_{-i}, p_{-i}), the objective function (*) and restrictions (1), (2), and (3) constitute a Linear Programming problem. Given (q^*_{-i}, p^*_{-i}), a pair (Q^*, P^*) is said to be a Nash Equilibrium if (q^*_i, p^*_i) is the solution of this Linear Programming problem for all $i\epsilon N$. In other words, (Q^*, P^*) is a Nash Equilibrium if $\Pi_i(Q^*, P^*) \geq \Pi_i(q_i, q^*_{-i}, p_i, p^*_{-i})$ for all pairs (q_i, p_i) and for all $i\epsilon N$. Given that the creation of a scientific collaboration link requires mutual consent of the two agents involved and that researchers can announce any p and q vectors they wish satisfying restrictions (1)-(3) (multidimensional strategy space), a huge coordination problem arises. As such, the game displays a multiplicity of Nash Equilibria where mutually beneficial links can be left aside[8]. This is solved if players are allowed to coordinate bilaterally. For this reason, refinements on Nash Equilibrium that allow for coalitional moves are usually applied to this kind of network-formation games[9]. The refinement we use is the Bilateral Equilibrium that is defined as follows:

Definition 1 *A pair (Q^*, P^*) is a Bilateral Equilibrium if the following conditions hold:*

- *(Q^*, P^*) constitutes a Nash equilibrium*
- *For any pair of players $i, j\epsilon N$ and every pair of strategies (q_i, p_i) and (q_j, p_j),*

$$\Pi_i(q_i, q_j, q^*_{-i-j}, p_i, p_j, p^*_{-i-j}) > \Pi_i(Q^*, P^*) \Rightarrow \Pi_j(q_i, q_j, q^*_{-i-j}, p_i, p_j, p^*_{-i-j}) < \Pi_j(Q^*, P^*).$$

[8]For example, a strategy profile in which $p_{ij} = q_{ij} = 0 \ \forall \ i \neq j$ (resulting in the empty network) is always a Nash Equilibrium.

[9]Contrarily to Bala and Goyal (2000) and others, our model presents directed links but both agents involved in a link benefit from its existence and mutual consent is required to form it; the direction of the link only refers to the flow of ideas. So, in spite of having directed links, we do not formulate the network formation as a non cooperative game.

We shall use the short term BE to refer to this concept. This notion of equilibrium is taken from Goyal and Vega-Redondo (2007); it generalizes the original formulation of pairwise stability due to Jackson and Wolinsky (1996) by allowing pairs of players to modify the intensity of their links simultaneously.

3 Results

Empirical studies such as Goyal, Van der Leij and Moraga (2006) or Newman (2004) show that scientific collaboration networks present a highly unequal and hierarchical distribution of links. This implies that some researchers can be much more active and productive than others and, consequently, they can enjoy a much better scientific reputation. One may think that big intrinsical differences among researchers can constitute the main driving force behind these huge inequalities. Nevertheless, this paper shows that this is not necessarily the case and highly unequal and hierarchical networks may naturally emerge from the strategic interaction among very similar (non-identical) researchers. In our equilibrium networks, some agents will be able to exhaust their processing capacity whereas some others will not process any idea at all. Propositions (1)-(5) present a list of necessary conditions that strongly narrow the set of potential BE networks. Throughout this section, we will illustrate the consequences of those conditions on the set of equilibrium networks. All proofs are relegated to the appendix.

The next result specifies the conditions that P^* must hold for any given Q.

Proposition 1 *For any given Q and for any $i \epsilon N$, p_i^* should satisfy:*

(i) $\rho \sum_{l \epsilon N} q_{li} p_{li}^* = \min\{\rho \sum_{l \epsilon N} q_{li}, 1\}$,

(ii) for any two agents $j, k \epsilon N_i$ such that $h_j > h_k$, it must be satisfied that $p_{ji}^* \geq p_{ki}^*$, and

(iii) $p_{li}^* \epsilon (0, 1)$ *for at most one agent $l \epsilon N_i$.*

For a given Q, agent i's best response p_i^* should maximize her payoff (*) and hold restrictions (1) and (3), i.e. p_i^* must be the solution of a Linear Programming problem[10]. As such, p_i^* must be one of the vertices of the polytope defined by restrictions (1) and (3). Conditions (i)-(iii) specify the characteristics of this vertex.

First, notice that agent i's payoff positively depends on p_{li} for any $l \epsilon N_i$. In consequence, any researcher should invest as much processing effort as possible (condition (i)). If $\rho \sum_{l \epsilon N} q_{li} \geq 1$ then

[10]It is easy to see that this solution is unique because $h_i \neq h_j$ for any pair $i, j \epsilon N$.

restriction (1) will be binding and agent i will exhaust their processing capacity. Otherwise, agent i will be able to invest the maximum effort to process each of the incoming ideas, i.e. $p^*_{li} = 1$ for all $l \epsilon N_i$. Proposition 1 offers two additional features of the optimal response p^*_i. In order to maximize the value of their contributions, researchers will preferably process the ideas coming from the most talented creators (condition (ii)) and they will invest as much effort as possible in processing those ideas; since $h_i \neq h_j$ for any pair $i, j \epsilon N$, condition (iii) follows. The next corollary summarizes the implications of Proposition 1. For $\rho \sum_{l \epsilon N} q_{li} > 1$, let $t \epsilon N_i$ be the least talented agent who holds $\rho \sum_{l \epsilon N: h_l > h_t} q_{li} \leq 1$.

Corollary 1 *For any $i \epsilon N$, p^*_i should satisfy:*

- *If $\rho \sum_{l \epsilon N} q_{li} \geq 1$, then $p^*_{li} = 1$ for all $l \epsilon N_i$ such that $h_l > h_t$, $p^*_{ti} = (1 - \rho \sum_{l \epsilon N: h_l > h_t} q_{li})/q_{ti}$, and $p^*_{ri} = 0$ for any other $r \epsilon N_i$.*
- *If $\rho \sum_{l \epsilon N} q_{li} < 1$, then $p^*_{li} = 1$ for all $l \epsilon N_i$.*

Given ρ and Q, notice that Corollary 1 determines the value of p^*_{ji} for all $i \epsilon N$ and for all $j \epsilon N_i$. Propositions (2)-(5) constitute a list of necessary conditions for Q^*. Next we present an immediate result:

Proposition 2 *For any pair of players $j, i \epsilon N$, q^*_{ji} can be positive only if $p^*_{ji} > 0$.*

Agents will not send their original ideas to those researchers who do not invest any effort in processing them. Since $\rho < 1$, these agents can always find an alternative processor for her ideas with some free processing capacity and both can profitably deviate by increasing the intensity of their collaboration relationship. Next we show that the best researchers will exhaust their processing capacity:

Proposition 3 *Whenever $q^*_{jk} > 0$ for some pair of players $j, k \epsilon N$, no player $i \epsilon N$ such that $i \neq j$ and $h_i > h_k$ can hold $\rho \sum_{l \epsilon N} q^*_{li} < 1$.*

In words, in equilibrium no agent $j \epsilon N$ should send part of their original ideas to some processor $k \epsilon N$ when a more talented agent $i \epsilon N$ does not exhaust their processing capacity. If that is the case, then i and j will be able to profitably deviate by increasing $q_{ji} p_{ji}$, *i.e.* the intensity of the scientific collaboration in which j and i are the creator and processor, respectively. By doing so, j benefits from the higher talent of i and i increases her expected number of contributions.

As a consequence of this proposition, in equilibrium the original ideas of our scientific community will accumulate on the most talented researchers. In order to illustrate the implications of Proposition 3, we define two subsets of N as follows:

$H^* = \{i\epsilon N : \ \rho \sum_{l\in N} q^*_{li} \geq 1\}$

$L^* = \{j\epsilon N : \ \rho \sum_{l\in N} q^*_{lj} < 1\}$[11].

Now we are able to write down the following statement:

Corollary 2 *For any two agents $i, j\epsilon N$ such that $i\epsilon H^*$ and $j\epsilon L^*$, h_i must be higher than h_j. The two most talented agents in L^* (say k and $k+1$) can have $q^*_{lk} > 0$ for some $l \in N$ and $q^*_{k,k+1} > 0$; $q^*_{l,k+1} = 0$ for all $l \neq k$, and $q^*_{lm} = 0$ for any $l \in N$ and any $m \in L^*$ such that $m \neq k, k+1$.*

Proposition 3 implies that any member of H^* is more talented that any agent in L^*. Notice, by contradiction, that if $\rho \sum_{l\in N} q^*_{li} < 1$ and $\rho \sum_{l\in N} q^*_{lj} \geq 1$ hold for some pair of agents $i, j \in N$ such that $h_i > h_j$, then there will exist some agent $r \in N$ such that $r \neq j$ and $q^*_{rj} > 0$. Thus, Proposition 3 would be violated.

A second implication of Proposition 3 is that almost all members of L^* will not receive any idea at all, *i.e.* $q^*_{lm} = 0$ for almost all $m \in L^*$. Notice that, by Proposition 3, if the second most talented agent in L^* (say k+1) has $q^*_{l,k+1} > 0$ for some l, then l cannot be different from the most talented agent in L^* (say k). We can follow the same reasoning to conclude that only the two most talented agents in L^* can receive original ideas from other researchers.

From this corollary we can conclude that our equilibrium candidates present a clear configuration. The population can be split into two subgroups: the most talented agents will receive ideas from others and they will be able to exhaust their processing capacity whereas the rest of researchers will not use all their capacity. In fact, only two agents in L^* can receive some idea to process. All the rest[12] will not receive any idea at all. Thus, our equilibrium candidates can present huge inequalities among researchers even though they are very similar; this result holds for any h, even for arbitrarily small (non-zero) differences among agents' talents.

Apart from being highly unequal, equilibrium networks present a strong hierarchy as illustrated by the following result:

Proposition 4 *If $q^*_{ki}p^*_{ki} > 0$ for some pair of agents $k, i\epsilon N$, then q^*_{jr} cannot be positive for any pair of agents $j, r\epsilon N$ such that (i) $j \neq i$, (ii) $h_j > h_k$, and (iii) $h_r < h_i$.*

[11] Since $\rho < 1$, L^* is nonempty. Notice also that for any given ρ, Proposition 3 implies that H^* is nonempty when n is sufficiently large.

[12] This can include a large number of agents if ρ is low and n is large.

In other words, whenever there is an active collaboration relationship between agents k and i in which k is the creator and i is the processor, any agent j (different from i) such that $h_j > h_k$ should send all their original ideas to agents with a talent higher or equal than h_i. Otherwise, agents i and j can profitably deviate by increasing the intensity of their collaboration relationship in which j is the creator and i is the processor. This result imposes a clear hierarchical structure on our equilibrium candidates, because it implies that the higher is the talent of a researcher the more talented are her collaborators. Notice again that this result holds for any arbitrarily small (non-zero) differences among agents' talents.

Thus, Propositions 3 and 4 already show that almost identical self-interested researchers will organize themselves forming hierarchical and unequal structures and we do not need to impose huge a priori differences among researchers to reproduce those collaboration structures (this contrasts with the results of van der Leij (06) in which a minimum degree of heterogeneity among agents is required in order to reproduce the empirical results about in-degree inequality); in the equilibrium networks of our model some agents (those in H^*) will enjoy a relatively large payoff because they will be able to exhaust their processing capacity whereas the rest of agents (those in L^*) will be much less productive because they cannot fully exploit their processing capacity. Moreover, the hierarchical structure announced above raises this inequality among equilibrium payoffs because the highly talented researchers will collaborate with each other and this increases the value of their contributions.

From the last two propositions, the next corollary follows:

Corollary 3 *(a) No agent i can have $q^*_{i,i+k} > 0$ for any natural number $k \geq 2$.*

(b) If $q^*_{ij} > 0$ and there exists some agent $l \in N_j$ such that $h_l < h_i$, then $q^*_{i,j+k}$ cannot be positive for any natural number k.

In other words, (a) no agent will send part of her original ideas to some researcher located two positions below her in the ranking of talents and (b) when agent i send original ideas to j and researcher i is not the least talented agent in N_j then she cannot send original ideas to any agent below j. With respect to (a), assume by contradiction that $q^*_{i,i+k} > 0$ for some natural number $k \geq 2$. If $\rho \sum_{l \in N} q^*_{l,i+1} < 1$, then Proposition 3 is violated. Consider now that $\rho \sum_{l \in N} q^*_{l,i+1} \geq 1$. Since $\rho < 1$, the first $i+1$ agents in the ranking of talents can fully process the original ideas generated by i researchers. In consequence, if $\rho \sum_{l \in N} q^*_{l,i+1} \geq 1$ and $q_{i,i+k} > 0$ for some $k \geq 2$ then one of these two cases must hold: (i) $\rho \sum_{l \in N} q^*_{l,j} < 1$ for some agent j such that $j \leq i$ or (ii) $q^*_{j,i+1} > 0$ for some agent j such that $j \geq i$. The first case violates Proposition 3 and the second case violates Proposition 4. With respect to (b), notice that whenever $i \neq j$ this statement is a direct consequence of Proposition 4. The case in which $i = j$ is explained in the appendix.

In order to offer further details of our equilibrium networks we will distinguish two subcases. First, let us consider that $\rho \leq \frac{i-1}{i}$, where i is a natural number. When $\rho \leq \frac{i-1}{i}$, the processing capacity of $i-1$ agents allows to process the original ideas created by i researchers (ρi); specifically, the $i-1$ researchers that precede agent i in the ranking of talents can fully process all original ideas generated by the first i researchers in that ranking[13]. As a consequence of Propositions (1)-(4) agent i will send all their original ideas to researchers with a higher talent. In such a case we can further detail our equilibrium candidates as follows:

Proposition 5 *Consider that $\rho \leq \frac{i-1}{i}$ and let j be the most talented agent in M_i. In equilibrium the following must hold:*

(i) $q^*_{i,j+k} = 0$ *for any natural number $k > 1$.*

(ii) $q^*_{i,j+1} = 0$, *if $\exists l \in N_j$ such that $h_l < h_i$.*

(iii) $q^*_{i,j+1} \epsilon \{0, \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj} - \frac{1-\rho}{\rho}\}$, *if $\nexists l \in N_j$ such that $h_l < h_i$.*

First of all notice that Proposition 2 implies that $q^*_{ik} > 0$ can only hold when p^*_{ik} is positive. Therefore, given that $\rho \leq \frac{i-1}{i}$, part (i) of Proposition 5 states that in equilibrium agent i's original ideas can have, at most, two different destinations that must be located at consecutive positions in the ranking of talents. In other words, any particular node can have at most two out-degree links[14]. Let j be the most talented destination of agent i's original ideas. Part (ii) states that whenever i is not the least talented agent in N_j, this destination must be unique, *i.e.* $q^*_{ij} = 1$. Part (iii) specifies the equilibrium values that q^*_{ij} can take when i is the least talented agent in N_j. Since there are two possible equilibrium values for $q^*_{i,j+1}$ we cannot fix a unique equilibrium network candidate. Nevertheless, Propositions (1)-(5) narrow this set to a great extent.

When $\rho > \frac{i-1}{i}$ the number of original ideas created by i agents (ρi) exceeds the processing capacity of $i-1$ agents. Specifically, the $i-1$ researchers that precede agent i in the ranking of talents cannot fully process all original ideas generated by the first i researchers in that ranking. In consequence, agent i will have to retain part of their original ideas or send them to agent $i+1$ (see Corollary 3). The optimal decision will depend on the comparison between $f(h_i, h_i)$ and $f(h_i, h_{i+1})$ which has not

[13] Notice that this condition cannot hold for $i = 1$.

[14] The number of in-degree links is not bounded.

been fixed by Assumption 1. Thus, we need to specify a function $f(\cdot)$ and a vector h in order to fully determine the destinations of agent i's original ideas.

In order to illustrate the implications of our results on the equilibrium candidates, we present the next example.

Example 1 *Let us assume that $n = 70$ and $\rho = 0.1$. Since agents' processing capacity is normalized to one, in this example agents must receive ideas from, at least, 10 different origins in order to exhaust their processing capacity. Propositions (2)-(4) restrict the possible equilibrium vectors q_i^* for all $i \in N$. Moreover, notice that we can apply Proposition (5) to all agents $i \in N$ except for $i = 1$. Thus, Proposition (2)-(5) almost determine Q^*. It only remains to fix q_1^* and the full specification of Q^* will follow. The most talented agent in N can either retain their own original ideas or send them to agent 2 depending on the comparison between $f(h_1, h_1)$ (*i.e.* the value of a contribution in which 1 creates and processes an original idea) and $f(h_1, h_2)$ (*i.e.* the value of a contribution in which 2 processes an original idea of agent 1). Let us focus on one of these two possibilities; say that agent 1 prefers to send their original ideas to 2. Thus, for any vector of talents (h) and for a broad family of $f(\cdot)$ functions (assumption 1 is the unique requirement) we can fully specify Q^* as follows:*

$q_i^* = (1, 0, 0, ..., 0)$ *for any agent* $i \in [2, 11]$,
$q_i^* = (0, 1, 0, 0, ..., 0)$ *for* $i = 1$ *and* $i \in [12, 20]$,
$q_i^* = (0, 0, 1, 0, ..., 0)$ *for any agent* $i \in [21, 30]$,
$q_i^* = (0, 0, 0, 1, 0, ..., 0)$ *for any agent* $i \in [31, 40]$,
and so on.

Applying Proposition 1 we can obtain P^*. The vectors that constitute this matrix are described below:

$$p_{i1}^* = \begin{cases} 1 & , \textit{if } i \in [2, 11]; \\ 0 & , \textit{otherwise.} \end{cases}$$

$$p_{i2}^* = \begin{cases} 1 & , \textit{if } i = 1 \textit{ or } i \in [12, 20]; \\ 0 & , \textit{otherwise.} \end{cases}$$

$$p_{i3}^* = \begin{cases} 1 & , \textit{if } i \in [21, 30]; \\ 0 & , \textit{otherwise.} \end{cases}$$

$$p_{i4}^* = \begin{cases} 1 & , \textit{if } i \in [31, 40]; \\ 0 & , \textit{otherwise.} \end{cases}$$

and so on.

The equilibrium network $G^* = P^* \otimes Q^*$ is illustrated in Figure 2.

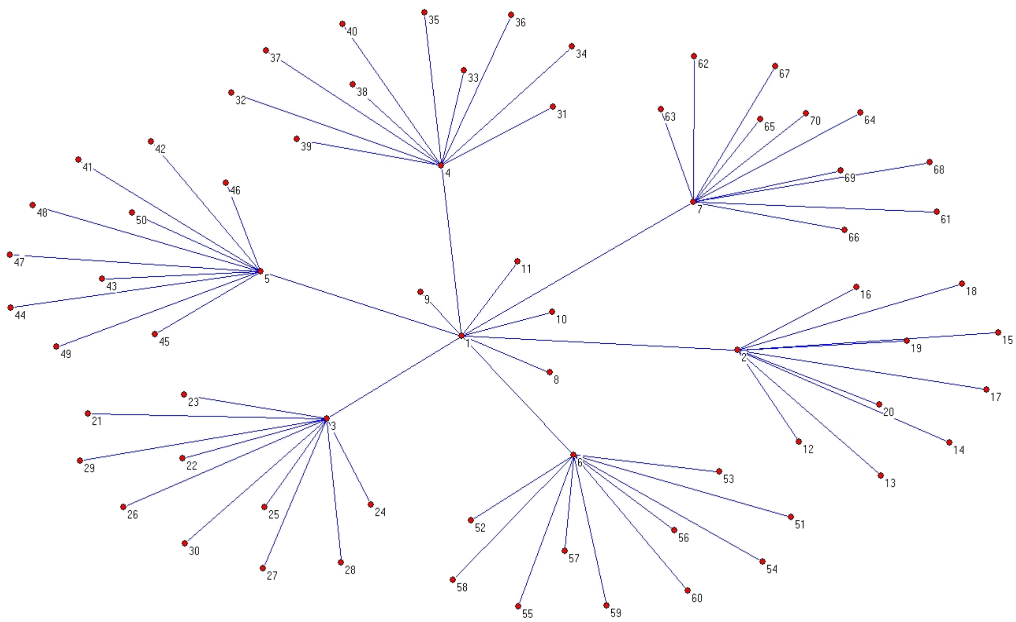

Figure 2: Example with n=70 and $\rho = 0.1$

Notice the resemblance of this network with the empirical observation represented in Figure 1. The main contribution of this paper is to show how highly unequal and hierarchical structures can arise from the strategic interaction among very similar (non-identical) researchers[15]*. As announced by Corollary 2, agents in our equilibrium network self-organize in two groups. In this example agents 1-7 constitute group H^* and the rest constitute group L^*. Those agents in H^* receive original ideas from others up to the point in which they can exhaust their processing capacity. Agents 8-70 do not receive any idea from others and they only send their ideas to one of the first seven researchers. Moreover, there is a strict order in the collaboration pattern; agents with a higher talent can send their original ideas to higher talented processors. As a consequence of these two effects, (i) agents 1-7 will enjoy a higher productivity than the rest of researchers and (ii) there can be a huge difference across agents in terms of the value of their contributions because higher talented agents can work with higher talented collaborators.*

[15] In our scientific collaboration network links are directed in the sense that ideas flow from one collaborator to the other. This direction is not represented in Figure 2 but is captured by the matrices P^* and Q^* described above.

4 Conclusion

In spite of the large body of empirical research about scientific collaboration networks, there is a lack of foundational theoretical models that analyze how individual decisions contribute to scientific collaboration network formation. This paper proposes a natural network formation game in which heterogeneity among researchers and limited processing and creative capacities drive the results. The model allows us to figure out the conditions under which self-interested researchers will organize themselves forming unequal and hierarchical scientific collaboration networks as is observed in the real-world without the necessity of imposing huge *a priori* differences among researchers.

References

Albert, R. and A.L. Barabási, 2002, "Statistical mechanics of complex networks", *American Physical Society*, 74, 1, 47-97

Arenas, A., A. Cabrales, L. Danon, A. Díaz-Aguilera, R. Guimerà, and F. Vega-Redondo, 2003, "Optimal information transmission in organizations: Search and congestion", *Working paper Universitat Pompeu Fabra*, 698.

Calvó-Armengol, A. and M.O. Jackson, 2004, "The effects of social networks on employment and inequality", *American Economic Review*, 94, 3, 426-454.

Cooper, C and A. Frieze, 2003, "A general model of web graphs", *Random Structures and Algorithms*, 22, 3, 311-335.

Goyal, S., M. van der Leij, and J.L. Moraga-González, 2006, "Economics: An emerging small world", *Journal of Political Economy*, 114, 2, 403-412.

Goyal, S. and S. Joshi, 2006, "Unequal Connections", *International Journal of Game Theory*, 34, 319-349.

Goyal, S. and F. Vega-Redondo, 2007, "Structural Holes in Social Networks", *Journal of Economic Theory*, 137, 460-492.

Goyal, S. and F. Vega-Redondo, 2007, "Structural Holes in Social Networks", *Journal of Economic Theory*, 137, 460-492.

Hudson, J., 1996, "Trends in multi-authored papers in Economics ", *Journal of Economic Perspectives*, 10, 3, 153-158.

Jackson, M.O. and A. Wolinsky, 1996, "A strategic model o economic and social networks", *Journal of Economic Theory*, 71, 1, 44-74.

Jackson, M.O. , 2004, "A survey of models of network formation: Stability and Efficiency", in *Group Formation in Economics; Networks, Clubs and Coalitions*, Cambridge Universtity Press.

Jackson, M.O. and B.W. Rogers, 2007, "Meeting strangers and friends of friends: How random are social networks?", *American Economic Review*, 97, 3, 890-915.

Kranton, R.E. and D. Minehart, 2001, "A Theory of Buyer-Seller Networks", *American Economic Revview*, 91, 3, 485-508.

Laband, D.N. and R.D. Tollison, 2000, "Intelectual collaboration", *Journal of Political Economy*, 108, 3, 632-662.

Milgram, S., 1967, "The small world problem", *Psycology today*, 2, 60-67.

Newman, M.E.J., 2001a, "Scientific collaboration networks I: Network construction and fundamental results", *Physical Review E.*, 64, 1.

Newman, M.E.J., 2001b, "Scientific collaboration networks II: Shortest paths, weighted networks, and centrality", *Physical Review E.*, 64, 1.

Newman, M.E.J., 2003, "The structure and function of complex networks", *SIAM Review*, 45, 2, 167-256.

Newman, M.E.J., 2004, "Co-authorship networks and patterns of scientific collaboration", *PNAS*, 101, suppl. 1, 5200-5205.

Price, D.J.d.S. , 1976, "A general theory of bibliometric and other cumulative advantage processes", *Journal of the American Society for Information Science*, 27, 292-306.

van der Leij, M., 2006, *The Economics of Networks: Theory and Empirics*, Ph.D. thesis, Tinbergen Institute. Erasmus University.

Watts, D., 1999, *Small Worlds*, Princeton Universtity Press.

A Proofs

Proof of Proposition 1. (i) Notice that restrictions (1) and (3) already imply

$$\rho \sum_{l \epsilon N} q_{li} p^*_{li} \leq \min\{\rho \sum_{l \epsilon N} q_{li}, 1\} \text{ for any } p_i.$$

By contradiction with the statement of the proposition, let us assume that $\rho \sum_{l \epsilon N} q_{li} p^*_{li} < \min\{\rho \sum_{l \epsilon N} q_{li}, 1\}$. This implies that for some $j \epsilon N_i$, there exists a $p'_{ji} > p^*_{ji}$ holding restrictions (1) and (3). Since agent i's payoff depends positively on p_{ji}, this agent can profitably deviate by choosing p'_{ji} instead of p^*_{ji}. Thus, the above inequality cannot hold in equilibrium. (ii) By contradiction let us assume that the equilibrium vector p^*_i holding restrictions (1) and (3) holds $p^*_{ki} > p^*_{ji}$ for some pair of players $j, k \epsilon N_i$, such that $h_j > h_k$. We claim that agent i can profitably deviate by choosing $p'_{ji} \epsilon (p^*_{ji}, 1]$ and $p'_{ki} \epsilon [0, p^*_{ki})$. To assure that restriction (3) still holds after the deviation the following must be satisfied:

$$(p'_{ji} - p^*_{ji}) q_{ji} = (p^*_{ki} - p'_{ki}) q_{ki}$$

It is easy to check that the pair

$$p'_{ji} = min\{1, p^*_{ji} + p^*_{ki} \frac{q_{ki}}{q_{ji}}\}$$

$$p'_{ki} = p^*_{ki} - (p'_{ji} - p^*_{ji}) \frac{q_{ji}}{q_{ki}}$$

satisfies the above conditions, and so restrictions (1)-(3). The marginal payoff for the deviator i is:

$$\Delta \Pi_i = \rho[(p'_{ji} - p^*_{ji}) q_{ji} f(h_j, h_i) + (p'_{ki} - p^*_{ki}) q_{ki} f(h_k, h_i)]$$

Using the definition of p'_{ki} we can write:

$$\Delta \Pi_i = \rho (p'_{ji} - p^*_{ji}) q_{ji} [f(h_j, h_i) - f(h_k, h_i)] > 0$$

Therefore $p^*_{ki} > p^*_{ji}$ cannot hold in equilibrium.

(iii) Assume by contradiction that $p^*_{ji}, p^*_{ki} \in (0, 1)$ for some pair of agents $j, k \in N_i$ such that $h_j > h_k$. The deviation considered in case (ii) is also possible and generates a positive marginal payoff to the deviator. In consequence, p^*_{ji} and p^*_{ki} cannot be between 0 and 1 in equilibrium. ■

Proof of Proposition 2. Assume by contradiction that $q^*_{ji} > 0$ and $p^*_{ji} = 0$. Since $\rho < 1$, there exists some agent $k \in N$ such that $\rho \sum_{l \in N} q^*_{lk} < 1$. By Proposition 1, $p^*_{lk} = 1 \; \forall l \in N_k$. Agents j and k can jointly deviate by choosing $p'_{jk} = 1$, $q'_{jk} \in (q^*_{jk}, 1]$, and $q'_{ji} \in [0, q^*_{ji})$. To satisfy restrictions (1)-(3) after the deviation, the following must hold:

$$q'_{jk} - q^*_{jk} = q^*_{ji} - q'_{ji}$$

$$1 - \rho \sum_{l \in N} q^*_{lk} \geq \rho(q'_{jk} - q^*_{jk})$$

By setting:

$$q'_{jk} = min\{1, q^*_{jk} + q^*_{ji}, q^*_{jk} + (\frac{1}{\rho} - \sum_{l \in N} q^*_{lk})\}$$

$$q'_{ji} = q^*_{ji} + (q'_{jk} - q^*_{jk})$$

the above conditions are satisfied. The deviators' marginal payoffs are:

$$\Delta\Pi_k = \Delta\Pi_j = \rho(q'_{jk} - q^*_{jk})f(h_j, h_k)$$

Since $q'_{jk} > q^*_{jk}$, we conclude that j and k will jointly deviate and q^*_{ji} cannot be positive when $p^*_{ji} = 0$.

■

Proof of Proposition 3. Consider by contradiction an equilibrium in which $q^*_{jk} > 0$ and $\rho \sum_{l \in N} q^*_{li} < 1$ for some agent $i \neq j$ such that $h_i > h_k$. Notice that, by restriction (2), $q^*_{jk} > 0$ implies that $q^*_{ji} < 1$. Notice also that, by Proposition 1, $p_{li} = 1$ for all $l \in N_i$. Let us consider that i and j jointly deviate by choosing $p'_{ji} = 1$, $q'_{ji} \in (q^*_{ji}, 1]$, and $q'_{jk} \in [0, q^*_{jk})$. In order to assure that restrictions (1)-(3) still hold after the deviation the following conditions must be satisfied:

$$q'_{ji} - q^*_{ji} = q^*_{jk} - q'_{jk}$$

$$1 - \rho \sum_{l \in N} q^*_{li} \geq \rho(q'_{ji} - q^*_{ji})$$

It is easy to check that the pair

$$q'_{ji} = min\{1, q^*_{ji} + q^*_{jk}, q^*_{ji} + (\frac{1}{\rho} - \sum_{l \in N} q^*_{li})\}$$

$$q'_{jk} = q^*_{jk} - (q'_{ji} - q^*_{ji})$$

satisfies the above conditions, and so restrictions (1)-(3). The marginal payoff for the deviators i and j can be written as follows:

$$\Delta\Pi_i = \rho(q'_{ji} - p^*_{ji}q^*_{ji})f(h_j, h_i)$$

$$\Delta\Pi_j = \rho[(q'_{ji} - p^*_{ji}q^*_{ji})f(h_j, h_i) - (q^*_{jk} - q'_{jk})p^*_{jk}f(h_j, h_k)]$$

Since $q'_{ji} > q^*_{ji}$ and $p^*_{ji} \leq 1$, we can conclude that $\Delta\Pi_i > 0$. With respect to $\Delta\Pi_j$, we can use the definition of q'_{jk} and some simple algebra to write:

$$\Delta\Pi_j \geq \rho(q'_{ji} - q^*_{ji})(f(h_j, h_i) - f(h_j, h_k))$$

Since $q'_{ji} > q^*_{ji}$, $h_i > h_k$, and $i \neq j$ we can conclude that $\Delta\Pi_j > 0$. In consequence, both agents will agree on that deviation and the situation introduced in the beginning of the proof cannot be hold in equilibrium. ∎

Proof of Proposition 4. Let us assume by contradiction that $q^*_{jr} > 0$ under conditions (i)-(iii) stated in the proposition. Notice that $q^*_{jr} > 0$ implies that $q^*_{ji} < 1$. Notice also that $p^*_{ki} > 0$. Let us consider that i and j jointly deviate by choosing $p'_{ji} = 1$, $q'_{ji} \in (q^*_{ji}, 1]$, $q'_{jr} \in [0, q^*_{jr})$, and $p'_{ki} \in [0, p^*_{ki})$. In order to assure that restrictions (1)-(3) still hold after the deviation the following conditions must hold:

$$q'_{ji} - q^*_{ji} = q^*_{jr} - q'_{jr}$$

$$q'_{ji} - p^*_{ji}q^*_{ji} = q^*_{ki}(p^*_{ki} - p'_{ki})$$

By Proposition 1, $q^*_{ki}p^*_{ki} > 0$ and $h_j > h_k$ imply that $p^*_{ji} = 1$ whenever $q^*_{ji} > 0$. Therefore, the second condition can be written as follows:

$$q'_{ji} - q^*_{ji} = q^*_{ki}(p^*_{ki} - p'_{ki})$$

It is easy to check that $p'_{ji} = 1$ and

$$q'_{ji} = min\{1, q^*_{ji} + q^*_{jr}, q^*_{ji} + q^*_{ki}p^*_{ki})\}$$

$$q'_{jr} = q^*_{jr} + (q'_{ji} - q^*_{ji})$$

$$p'_{ki} = p^*_{ki} - \frac{q'_{ji} - q^*_{ji}}{q^*_{ki}}$$

are well defined and satisfy the above conditions. The marginal payoff for the deviators can be written as follows.

$$\Delta\Pi_i = \rho[(q'_{ji} - p^*_{ji}q^*_{ji})f(h_j, h_i) + q^*_{ki}(p'_{ki} - p^*_{ki})f(h_k, h_i)]$$

Substituting the last condition stated above we obtain:

$$\Delta\Pi_i = \rho q^*_{ki}(p'_{ki} - p^*_{ki})(f(h_j, h_i) - f(h_k, h_i))$$

Since $p^*_{ki} > p'_{ki}$, $h_i \neq h_j$, and $h_j > h_k$ then $\Delta\Pi_i > 0$. On the other hand,

$$\Delta\Pi_j = \rho[(q'_{ji} - p^*_{ji}q^*_{ji})f(h_j, h_i) + p^*_{jr}(q'_{jr} - q^*_{jr})f(h_j, h_r)]$$

Using the previous conditions we obtain:

$$\Delta\Pi_j = \rho(q^*_{jr} - q'_{jr})(f(h_j, h_i) - p^*_{jr}f(h_j, h_r))$$

Since $q^*_{jr} > q'_{jr}$, $h_i \neq h_j$, and $p_{jr} \leq 1$, then $\Delta\Pi_j > 0$. In consequence, both agents will deviate and the original situation cannot hold in equilibrium. ■

Corollary 3 (b). By statement (a) we can conclude that i is the most talented researcher who can send their original ideas to agent $i+1$ in equilibrium. By Proposition 1, $p^*_{i,i+1} = 1$ for any positive $q_{i,i+1}$. In consequence, $q^*_{ij} > 0$ for $i = j$ implies that agent i prefers to retain her own original ideas rather than sending them to agent $i+1$. Thus, if $q^*_{ii} > 0$ then $q^*_{i,i+k}$ cannot be positive for any natural number k. ■

Proof of Proposition 5. First notice that $\rho \leq \frac{i-1}{i}$ and Propositions (1)-(4) imply that $h_l > h_i$ for all $l \in M_i$. Let j be the most talented agent in M_i. By Corollary 3 (b), if there exists some $l \in N_j$ such that $h_l < h_i$ then q^*_{ij} must be one. Thus, part (ii) of Proposition 5 follows. In consequence, only if i is the least talented agent in N_j, then $q^*_{i,j+k}$ can be positive for some natural number k. Next, we will proof part (i).

Assume by contradiction that $q^*_{i,j+k} > 0$ for some natural number $k > 1$. Proposition 3 implies that $\rho \sum_{l \in N} q^*_{l,j+1} \geq 1$. Since $\rho < 1$, agent $j+1$ must receive original ideas from at least two different origins. At this point we must consider two possibilities:

(a) $\exists l \in N_{j+1}$ such that $h_l < hi$.

In this case, if $q^*_{i,j+k} > 0$ for some $k > 1$ then Proposition 4 is violated and we reach the desired contradiction.

(b) $h_l \geq h_i$, for all $l \in N_{j+1}$.

We claim that such a case cannot be sustained in equilibrium when $\rho \leq \frac{i-1}{i}$. Since i is the least talented agent in N_j, Proposition 4 implies that no agent with a talent higher than h_i, except for j, can be included in N_{j+1}. On the other hand, $\rho < 1$ and $\rho \sum_{l \in N} q^*_{l,j+1} \geq 1$ imply that N_{j+1} includes at least two different agents. In consequence, in this case only agents j and i are included in N_{j+1}. This has several implications. First, notice that j must be the least talented agent in N_{j-1}. In consequence, j is the best possible destination for the ideas of $j+1$. Moreover, since $j \in N_{j+1}$ and $\rho < 1$, agent j has some available capacity to process agent $j+1$'s original ideas. Second, $\rho \sum_{l \in N} q^*_{l,j+1} \geq 1$ implies that $\rho \geq \frac{1}{2}$. In consequence, applying Proposition 1 we can say that j can only receive original ideas from two different origins, which are $j+1$ and i. To hold Proposition 4 these two agents must be consecutive, *i.e.* $j+2 = i$. Therefore, a positive $q^*_{i,j+k}$ for $k > 1$ contradicts that $h_l > h_i$ for all $l \in M_i$. Therefore, case (b) is not sustainable when $\rho \leq \frac{i-1}{i}$.

Next, we prove part (iii). Let us assume by contradiction that $q^*_{i,j+1} > 0$ and $q^*_{i,j+1} \neq \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj} - \frac{1-\rho}{\rho}$.

As showed above in case (b), there must exist some agent $l \in N_{j+1}$ such that $h_l < hi$. In consequence, Proposition 1 implies that if $q^*_{i,j+1} > 0$ then $p^*_{i,j+1}$ cannot be different from 1. On the other hand, by Corollary 2 notice that $q^*_{i,j+1} > 0$ implies that $j \in H^*$, *i.e.* $\rho \sum_{l \in N} q^*_{lj} \geq 1$. Thus, by Proposition 1, $\rho \sum_{l \in N} q^*_{lj} p^*_{lj} = 1$. Since $q^*_{ij} p^*_{ij} > 0$, we can write:

$$q^*_{ij} p^*_{ij} = \frac{1}{\rho} - \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj}$$

Let us now consider two alternative cases:

- If $q^*_{ij} > \frac{1}{\rho} - \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj}$, then Proposition 1 implies that p^*_{ij} must be lower than 1. In this case, i and j can profitably deviate by choosing $p'_{ij} = 1$, $q'_{ij} = \frac{1}{\rho} - \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj}$ and $q'_{i,j+1} = q^*_{i,j+1} + (q'_{i,j} - q^*_{i,j})$. The marginal payoffs are:

$$\Delta \Pi_i = \rho [(q'_{i,j+1} - q^*_{i,j+1}) f(h_i, h_{j+1}) > 0$$

$$\Delta \Pi_j = 0$$

  In consequence, such a q^*_{ij} cannot hold in equilibrium.

- If $q^*_{ij} < \frac{1}{\rho} - \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj}$, then $\rho \sum_{l \in N} q^*_{lj} p^*_{lj} = 1$ cannot hold.

Therefore, whenever $q^*_{i,j+1}$ is positive then $q^*_{ij} = \frac{1}{\rho} - \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj}$ must hold and, in consequence, $p^*_{ij} = 1$. Since $q^*_{im} = 0$ for any $m \neq j, j+1$, restriction (2) implies that $q^*_{ij} + q^*_{i,j+1} = 1$. Thus, we can express the result in terms of $q^*_{i,j+1}$ and say: whenever $q^*_{i,j+1}$ is positive then $q^*_{i,j+1} = \sum_{l \epsilon N \setminus i} q^*_{lj} p^*_{lj} - \frac{1-\rho}{\rho}$ must hold. ■