The KPSS test with two structural breaks

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July 22, 2005

Abstract

In this paper we generalize the KPSS-type test to allow for two structural breaks. Seven models have been defined depending on the way that the structural breaks affect the time series behaviour. The paper derives the limit distribution of the test both under the null and the alternative hypotheses and conducts a set of simulation experiments to analyse the performance in finite samples.

Keywords: Stationary tests, structural breaks, unit root

JEL classification: C12, C15, C22

1 Introduction

Testing for unit roots in time series has become a usual practice in economic research. Since the seminal paper of Dickey and Fuller (1979) appeared, there have been multiple developments aimed at getting statistical tools consistent with seeming data generation process of macroeconomic time series. One of these developments has emerged after studying the effects that structural breaks can cause on the integration order analysis. In this regard, Perron (1989, 1994) shows that misspecification errors in the trend function involve a bias that make

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standard unit root tests application be risky, since they tend to favour the unit root hypothesis even when the time series is stationary. Early work of Perron (1989) has been extended in Banerjee, Lumsdaine, and Stock (1992), Perron and Vogelsang (1992), Zivot and Andrews (1992), Perron (1994, 1997), Clemente, Montañés, and Reyes (1998), and Montañés and Reyes (1998), among others. Briefly speaking, these authors generalise the analysis of Perron (1989, 1990) through the specification of several estimation methods for the selection of the date of the break.

Although the order of integration of time series is usually tested using unit root tests, some authors suggest to reverse the hypotheses and, hence, proceed to test the stationarity null hypothesis against the unit root alternative hypothesis as a way to establish the robustness of the unit root tests conclusions –see Park (1990), Amano and Van Norden (1992), Maddala and Kim (1998), and Carrion-i-Silvestre, Sansó, and Artís (2001). Park (1990), Tanaka (1990, 1995), Kwiatkowski, Phillips, Schmidt, and Shin (1992), Saikkonen and Luukkonen (1993), Leybourne and McCabe (1994, 1999) and Shin (1994) are some theoretical developments where stationarity tests are given. Unfortunately, these stationarity tests are not independent of structural breaks. Lee, Huang, and Shin (1997) have shown the inconsistency of the test in Kwiatkowski, Phillips, Schmidt, and Shin (1992) –hereafter KPSS test– when structural breaks are not taken into account. Similar results are achieved for the Leybourne and McCabe (1994) test. Busetti and Harvey (2001), Lee and Strazicich (2001) and Kurozumi (2002) generalize the KPSS-type test to accommodate the presence of one shift in the level and/or the slope of the time series. Busetti and Harvey (2001) also compute a small set of critical values for the KPSS test with two structural breaks for non trended variables and propose a simplified test that allows for up to four structural breaks, but only for two different deterministic specifications –they allow for level shifts for non trended variables and for both level and slope shifts for trended variables.

In this paper we extend the KPSS test to the case where there exist two structural breaks that change the level and/or the slope of the time series. Our proposal adds to the recent developments in the literature where the order of integration analysis allows for two structural breaks through the specification of up to seven different deterministic models –see Lumsdaine and Papell (1997), Clemente, Montañés, and Reyes (1998) and Lee and Strazicich (2003), among others, for the unit root test counterpart.

The paper is organized as follows. Section 2 presents the model and derives the limit distribution of the test statistic assuming that the structural breaks are known. Section 3 addresses the estimation of the date of the breaks. Section 4 looks into the finite sample performance of the test. Finally, some concluding remarks are made. Mathematical proofs are given in the Appendix.

Table 1: Deterministic specifications

Model	$f(t, T_{b1}, T_{b2})$
AAn	$\mu + \sum_{i=1}^{2} \theta_i \ DU_{i,t}$
AA	$\mu + \beta \ t + \sum_{i=1}^{2} \theta_i \ DU_{i,t}$
BB	$\mu + \beta \ t + \sum_{i=1}^{2} \gamma_i \ DT^{*}_{i,t}$
CC	$\mu + \beta \ t + \sum_{i=1}^{2} \theta_i \ DU_{i,t} + \sum_{i=1}^{2} \gamma_i \ DT^{*}_{i,t}$
AB-BA	$\mu + \beta \ t + \theta_1 \ DU_{1,t} + \gamma_2 \ DT^{*}_{2,t}$
AC-CA	$\mu + \beta \ t + \sum_{i=1}^{2} \theta_i \ DU_{i,t} + \gamma_2 \ DT^{*}_{2,t}$
BC-CB	$\mu + \beta \ t + \theta_2 \ DU_{2,t} + \sum_{i=1}^{2} \gamma_i \ DT^{*}_{i,t}$

2 The KPSS test with two breaks

Following Kwiatkowski, Phillips, Schmidt, and Shin (1992), we propose a model of unobserved components to test for stationarity against the unit root process when there might be two structural breaks affecting the trend function of the time series. The standard KPSS test is based on:

$$\begin{aligned} y_t &= f(t, T_{b1}, T_{b2}) + r_t + \varepsilon_t; && (1) \\ r_t &= r_{t-1} + u_t, && (2) \end{aligned}$$

where $u_t \sim iid\left(0, \sigma_u^2\right)$ and $\{\varepsilon_t\}$ is assumed to satisfy the strong-mixing regularity conditions of Phillips (1987) and Phillips and Perron (1988). Under the null hypothesis of stationarity σ_u^2 must be zero, otherwise the stochastic process is I(1). $f(t, T_{b1}, T_{b2})$ in (1) denotes the deterministic specification that is assumed for the time series. Table 1 presents the seven deterministic specifications that are considered in the paper. In order to take into account the presence of the structural breaks, these specifications incorporate dummy variables, which are defined as $DU_{i,t} = 1$, $DT^{*}_{i,t} = (t - T_{bi})$ if $t > T_{bi}$ and 0 otherwise, with $T_{bi} = \lambda_i T$, $\lambda_i \in (0, 1)$, $i = 1, 2$, denoting the date of the structural breaks.

The *pseudo* LM test is given by

$$\hat{\eta}_j = \hat{\sigma}^{-2} T^{-2} \sum_{t=1}^{T} S_t^2, \tag{3}$$

$j = \{AAn,\ AA,\ BB,\ CC,\ AB\text{-}BA,\ AC\text{-}CA,\ BC\text{-}CB\}$, where $S_t = \sum_{j=1}^{t} \hat{e}_j$, $S_0 = 0$, with $\hat{e}_t$ being the OLS estimated residuals of the regression of y_t on one of the deterministic specifications in Table 1. Kwiatkowski, Phillips, Schmidt, and Shin (1992) estimate the long-run variance from

$$\hat{\sigma}^2 = T^{-1} \sum_{t=1}^{T} \hat{e}_t^2 + 2T^{-1} \sum_{s=1}^{l} w(s, l) \sum_{t=s+1}^{T} \hat{e}_t \hat{e}_{t-s}. \tag{4}$$

where $w(s,l)$ denotes the spectral window –*i.e.* either the Bartlett or the Quadratic spectral windows. While the choice of the kernel is, to some extent, somewhat that depends on the preference of practitioners –Kwiatkowski, Phillips, Schmidt, and Shin (1992) use the Bartlett kernel, but Hobijn, Franses, and Ooms (1998) suggest the Quadratic spectral window–, some cautions should be taken when selecting the spectral bandwidth. Thus, we can find in the literature some suggestions that can drive to wrong conclusions. For instance, Lee (1996) uses Andrews (1991) method, while Hobijn, Franses, and Ooms (1998) suggest to apply the automatic methods in Newey and West (1994) to estimate the bandwidth. Unfortunately, Choi and Ahn (1995, 1999) and Kurozumi (2002) advice that the use of these data based selection methods provoke the inconsistency of the test. Notwithstanding, some bounds to control the estimated bandwidth can be imposed to avoid such inconsistence. Carrion-i-Silvestre and Sansó (2005a) have recently compared the different procedures to establish a bound for the bandwidth showing that the proposal in Sul, Phillips, and Choi (2003) is the best one in terms of size and power. Therefore, in this paper we use this estimator for the long-run variance. In brief, Sul, Phillips, and Choi (2003) propose a prewithened Heteroskedasticity and Autocorrelation Consistent (HAC) estimator for the long-run variance. In the first stage an AR model for the residuals $\{\hat{e}_t\}$ is estimated:

$$\hat{e}_t = \vartheta_1 \hat{e}_{t-1} + \ldots + \vartheta_p \hat{e}_{t-p} + \psi_t. \tag{5}$$

After the estimation of (5) is carried out it is possible to obtain the long-run variance of the estimated residuals in (5), which is denoted as $\tilde{\sigma}_\psi^2$, through the application of a HAC estimator –for instance, Bartlett or Quadratic Spectral window– to control for the presence of heteroskedasticity. In the second stage the estimated long-run variance is recolored:

$$\hat{\sigma}^2 = \frac{\tilde{\sigma}_\psi^2}{\tilde{\vartheta}(1)^2},$$

where $\tilde{\vartheta}(1)$ denotes the autoregressive polynomial $\tilde{\vartheta}(L) = 1 - \tilde{\vartheta}_1 L - \ldots - \tilde{\vartheta}_p L^p$ estimated in (5) evaluated at $L = 1$. In order to avoid the inconsistence of the test statistic, Sul, Phillips, and Choi (2003) suggest using the following boundary condition rule to obtain the long-run variance estimate:

$$\hat{\sigma}^2 = \min\left\{ \kappa T \tilde{\sigma}_\psi^2, \frac{\tilde{\sigma}_\psi^2}{\tilde{\vartheta}(1)^2} \right\},$$

where $\kappa > 0$ is a constant to be determined below. Sul, Phillips, and Choi (2003) use $\kappa = 1$. However, in order to improve the power of the test with a good size other values of κ might be suitable. Note that this does not modify the asymptotic properties of the estimator, because this boundary rule only acts under the alternative hypothesis and the rate of divergence is not affected. The application of this rule ensures that the estimated long-run variance is bounded above by $\kappa T \tilde{\sigma}_\psi^2$.

The following Theorem presents the limit distribution of the KPSS test with two structural breaks assuming that the date of the breaks are known. Section 3 deals with the procedure that can be applied in order to estimate the breaking dates.

Theorem 1 *Let $\{y_t\}$ be a stochastic process described by (1) and (2), with $\sigma_u^2 = 0$ and $\{\varepsilon_t\}$ satisfying the strong-mixing regularity conditions of Phillips (1987) and Phillips and Perron (1988). Let $T_{bi} = \lambda_i T$, $i = 1, 2$, $0 < \lambda_1 < \lambda_2 < 1$. Furthermore, as $T \to \infty$, $T_{bi} \to \infty$ so that λ_i remains constant, $i = 1, 2$. Then as $T \to \infty$:*

$$\hat{\eta}_j \Rightarrow \int_0^{\lambda_1} H_{1,j}^2 (r, \lambda_1, \lambda_2)\, dr + \int_{\lambda_1}^{\lambda_2} H_{2,j}^2 (r, \lambda_1, \lambda_2)\, dr + \int_{\lambda_2}^{1} H_{3,j}^2 (r, \lambda_1, \lambda_2)\, dr,$$

j = {AAn, AA, BB, CC, AB-BA, AC-CA, BC-CB} where $\Rightarrow$ denotes weak convergence to the associated probability measure and $H_{k,j}(\bullet)$,[1] $k = 1, 2, 3$, are complex functions of Wiener processes and of the break fractions that are shown in the Appendix.

The proof of Theorem 1 is outlined in the Appendix. Notice that the limit distributions distinguish between the three subperiods that have been defined by the two breaks. Asymptotic critical values for the models AAn, AA, BB, CC, AB-BA, AC-CA, BC-CB are collected in Tables 2 to 8 for different values of the break fraction parameters at the 1, 2.5, 5 and 10% signification level.

Note the symmetry that seems to be present in the tabulated critical values for those models with the same effect for the two structural breaks. If we take a look on each diagonal of the tables of critical values we can see that there exist a symmetric behaviour that depends on both the distance between the two break fractions and the distance that separate these values from the beginning (end) of the time period. For instance, in the model CC the critical value for the couple $(\lambda_1, \lambda_2) = (0.1, 0.2)$ is 0.0972 while for $(\lambda_1, \lambda_2) = (0.8, 0.9)$ it is 0.0966.

Once the limit distribution of the KPSS test with two structural breaks has been derived for the different models, now is time to assess the consistency of the test under the alternative hypothesis of non-stationarity. As in Lee and Strazicich (2001), the goal is to proof that under the alternative hypothesis the sum of the square of the partial processes $T^{-2} \sum_{t=1}^{T} S_t^2$ is $O_p(T^2)$ and the estimate of the long-run variance is $O_p(T)$, so that the test statistic is $O_p(T)$. Theorem 2 presents the limit distribution of the KPSS test under the alternative hypothesis.

Theorem 2 *Let $\{y_t\}$ be a stochastic process with DGP defined by (1) and (2), with $\sigma_u^2 > 0$ and $\{\varepsilon_t\}$ satisfying the strong-mixing regularity conditions of Phillips (1987) and Phillips and Perron (1988). Then as $T \to \infty$,*

$$\hat{\eta}_j (l) = O_p (T),$$

j = {AAn, AA, BB, CC, AB-BA, AC-CA, BC-CB}.

The proof is outlined in the Appendix.

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3 The estimation of the breaking points

Up to now we have assumed that the date of the breaks are known, an unrealistic situation in applied research. Besides, this was the criticism that received the early work of Perron (1989), since the order of integration analysis became conditioned on this *a priori* selection. The way to overcome this drawback consists on proceed to the estimation of the break points instead of assuming them as exogenous. Thus, Hao (1996), Busetti and Harvey (2001) and Lee and Strazicich (2001) apply the minimum functional to the sequence that results from the computation of the KPSS test for all possible break points. The argument that minimizes this sequence is taken as the estimate of the breaking point. Instead, Carrion-i-Silvestre and Sansó (2005b) propose the minimization of the sequence of sum of squared residuals (SSR) to estimate the date of the break and compare both procedures concluding in favour of the later. The minimization of the SSR has been also suggested in Kurozumi (2002). Therefore, in this paper we propose the application of the procedure in Bai and Perron (1998) that computes the global minimization of the SSR and choose as the estimate of the dates of the breaks the argument that minimizes the sequence of $SSR\left(T_{b1}, T_{b2}\right)$, where the SSR is obtained from the regression of $y_t = f\left(t, T_{b1}, T_{b2}\right) + e_t$, where $f\left(t, T_{b1}, T_{b2}\right)$ denotes one of the deterministic components in Table 1. Thus, the break points are estimated as:

$$\left(\hat{T}_{b1}, \hat{T}_{b2}\right) = \arg\min{}_{T_{b,1}, T_{b,2}} SSR\left(T_{b1}, T_{b2}\right).$$

Bai (1994, 1997) shows the T-consistency of the estimation of the break fraction parameter when it is estimated using this criteria for the case of one structural break. This result is extended in Bai and Perron (1998) for the case of multiple breaks for both trending and non trending regressors. The same applies in this case. Notice that some trimming is required when computing estimates of the break points. Though the amount of trimming is somewhat arbitrary some practitioners have specified $\lambda_i \in [0.15, 0.85]$, $i = 1, 2$ –see among others Zivot and Andrews (1992).

Thus and provided that a T-consistent estimation of the breaking fractions is available, the KPSS test can be computed as usual and compared with the critical values collected in Tables 2 to 8 in order to test the null hypothesis of stationarity.

4 A simple Monte Carlo experiment

The performance of the test statistic in finite samples is carried out through a Monte Carlo experiment where both the empirical size and power are evaluated. The three different sets for the DGP parameters that are considered allow us to analyse the magnitude of the breaks influence. Our main interest points to the model CC since it is the most general formulation considered in this paper.

The DGP is given by

$$\begin{aligned} y_t &= f(t, T_{b1}, T_{b2}) + \varepsilon_t \\ \varepsilon_t &= \rho\, \varepsilon_{t-1} + v_t, \end{aligned} \quad (6)$$

with $f(t, T_{b1}, T_{b2})$ being the one for model CC, and $v_t \sim iid\ N(0,1)$. We have essayed four different vectors of parameters $\vartheta = (\mu, \beta, \theta_1, \gamma_1, \theta_2, \gamma_2)$: (i) $\vartheta_1 =$ (0, 0, 0, 0, 0, 0), (ii) $\vartheta_2 =$ (1, 0.2, -5, -0.2, 2, 0.5), (iii) $\vartheta_3 =$ (1, 0.2, 0, 0, -5, -0.2), and (iv) $\vartheta_4 =$ (1, 0.2, 0, 0, 0, 0), for the break points located at (λ_1, λ_2) = (0.2, 0.6). Simulations not reported here indicate that similar results were obtained for different specifications of the break points –results for (λ_1, λ_2) = (0.4, 0.7) are available upon request. The definition of $\rho = \{0.8, 0.9, 1\}$ in (6) allows us to analyse both the empirical size and power of the test statistic. The sample sizes are set at $T = \{100,\ 200,\ 500\}$, and $n = 2,000$ replications are conducted.

The estimation of the long-run variance is obtained using the Quadratic spectral windows with the automatic lag length selection method described in Sul, Phillips, and Choi (2003) along with the definition of the boundary condition depends on κ. Non reported Monte Carlo experiments led us to fix $\kappa = 0.15$ as a compromise between size and power, although, as mentioned above, the modification of the boundary condition does not affect neither the asymptotic size nor the asymptotic power of the test statistic, but allows to increase the power performance of the statistic in finite samples. All the computations are carried out using a GAUSS code available from the authors upon request. Simulation results are reported in Table 9, from which we can conclude that, in general, the empirical size of the statistic is close to the nominal one both for $\rho = 0.8$ and $\rho = 0.9$. Note that in these cases the large autoregressive parameter implies high persistence in the residuals, but the test still shows good empirical size. In contrast, the use of other boundary rules does not warrant controlled empirical size. For instance, the rule defined in Kurozumi (2002) for the KPSS statistic with one structural break shows size distortion problems for $\rho = 0.8$ –see Carrion-i-Silvestre and Sansó (2005a) for a comparison on the performance of different boundary rules for the case of no structural break. This conclusion is obtained irrespective of vector of parameters ϑ_i, $i = 1, \ldots, 4$. Thus, even for the case of no structural breaks the empirical size maintains at resonable level –see results for ϑ_1 and ϑ_4 in Table 9.

As expected, empirical power depends both on the sample size and on the magnitude of the structural breaks. As can be seen from Table 9, the power is higher for ϑ_2 and ϑ_3 sets of parameters than for the ϑ_1 and ϑ_4 ones. Furthermore, the test offers higher power when there are two structural breaks (ϑ_2) than when there is only one structural break (ϑ_3). In all, the finite sample performance of the stationarity tests with two structural breaks has revealed that the statistics that have been proposed in this paper have good statistical properties.

5 Conclusions

In this paper we have extended the standard stationarity test statistic of Kwiatkowski, Phillips, Schmidt, and Shin (1992) to allow for two structural breaks. Our proposal complements that of Lumsdaine and Papell (1997) and Clemente, Montañés, and Reyes (1998) for the DF unit root test and serves as a statistical instrument that can be used in confirmatory analysis.

Once the formalization of the different models we deal with, the asymptotic distribution of the test statistics has been derived both under the null and under the alternative hypotheses. In this way, the consistency of the test statistics has been proved. After obtaining the asymptotic distribution under the null hypothesis we have computed suitable critical values for different combinations of couples of break fractions. Critical values tabulation has been done assuming that the structural breaks are known *a priori* by the analyst. Moreover, these critical values can also be applied when we proceed to the estimation of the dates of the breaks through the minimisation of the SSR. Small simulation experiment to assess the finite sample size of the test statistics has been conducted, which shows that the statistics that we propose have desired properties in terms of empirical size and power.

6 Appendix. Asymptotic distributions and consistency

6.1 Asymptotic distributions

In this appendix we present the way we have obtained the asymptotic distributions for the model CC, which is the most general one. The proof for the other models follows the one described here and can be found in Carrion-i-Silvestre (1999). The estimated residuals that are used to build the test statistic are obtained from $\hat{e}_t = \varepsilon_t - z_t P \left(P z' z P\right)^{-1} \; P z' \varepsilon$, where P is a diagonal scaling matrix, $P = diag(T^{-1/2}, T^{-3/2}, T^{-1/2}, T^{-3/2}, T^{-1/2}, T^{-3/2})$, z denotes the regressors' matrix and ε is the vector of disturbances. In the limit, the cross-product symmetric matrix of regressors tends to $Pz'zP \rightarrow A$, with elements $A[1,1] = 1$, $A[1,2] = (1/2)$, $A[1,3] = (1-\lambda_1)$, $A[1,4] = (1-\lambda_1)^2/2$, $A[1,5] = (1-\lambda_2)$, $A[1,6] = (1-\lambda_2)^2/2$, $A[2,2] = 1/3$, $A[2,3] = \left(1-\lambda_1^2\right)/2$, $A[2,4] = \left(2-3\lambda_1+\lambda_1^3\right)^2/6$, $A[2,5] = \left(1-\lambda_2^2\right)/2$, $A[2,6] = \left(2-3\lambda_2+\lambda_2^3\right)^2/6$, $A[3,3] = (1-\lambda_1)$, $A[3,4] = (1-\lambda_1)^2/2$, $A[3,5] = (1-\lambda_2)$, $A[3,6] = (1-\lambda_2)^2/2$, $A[4,4] = (1-\lambda_1)^3/3$, $A[4,5] = (1-2\lambda_1+\lambda_2)(1-\lambda_2)/2$, $A[4,6] = (2-3\lambda_1+\lambda_2)(1-\lambda_2)^2/2$, $A[5,5] = (1-\lambda_2)$, $A[5,6] = (1-\lambda_2)^2/2$ and $A[6,6] = (1-\lambda_2)^3/3$. Following Phillips (1987) and Schmidt and Phillips (1992), the term involving the disturbances $Pz'\varepsilon \Rightarrow \sigma B$, where $B[1,1] = W(1)$, $B[2,1] = W(1) - \int_0^1 W(r)\,dr$, $B[3,1] = W(1) - W(\lambda_1)$, $B[4,1] = (1-\lambda_1)(W(1) - W(\lambda_1)) - \int_0^1 W(r)\,dr$, $B[5,1] = W(1) - W(\lambda_2)$ and $B[6,1] = (1-\lambda_2)(W(1) - W(\lambda_2)) - \int_0^1 W(r)\,dr$. Because of the consideration of two structural breaks in the deterministic component we have to distinguish between three different subperiods -in general, if we assume that there might be m-structural breaks we will be able to differentiate between $(m+1)$ subperiods. Therefore, for $t \leq T_{b1}$ the partial sum of the regressors converges to $T^{-1/2}\sum_{i=1}^{[rT]} z_i P \Rightarrow \sigma \left[\begin{array}{cccccc} r & r^2/2 & 0 & 0 & 0 & 0 \end{array} \right]$, whereas the partial sum processes of the estimated residuals converges to

$$T^{-1/2}S_t \Rightarrow \sigma \left(W(r) - \left[\begin{array}{cccccc} r & r^2/2 & 0 & 0 & 0 & 0 \end{array} \right] A^{-1}B \right).$$

After some cumbersome algebra manipulations we obtain that

$$\begin{aligned} T^{-1/2}S_t &\Rightarrow \frac{\sigma}{\lambda_1^3}\left[-4\lambda_1^2 r W(r) - 6r^2 W(\lambda_1) + 3\lambda_1 r(2+r) W(\lambda_1) + \lambda_1^3 W(r)\right] \\ &= \sigma H_{1,CC}(r, \lambda_1, \lambda_2). \end{aligned}$$

For $T_{b1} < t \leq T_{b2}$, the partial sum processes converges to

$$\begin{aligned} T^{-1/2}S_t \Rightarrow & \frac{\sigma}{(\lambda_1 - \lambda_2)^3}\left[3\lambda_2 r(2+r)(W(\lambda_1) - W(\lambda_2)) + 4\lambda_2^2 r(W(\lambda_1) - W(\lambda_2))\right. \\ & +6r^2(W(\lambda_2) - W(\lambda_1)) + \lambda_2^1(4r(W(\lambda_2) - W(\lambda_1)) + \\ & \lambda_2(4W(\lambda_1) - W(\lambda_2) - 3W(r))) \\ & +\lambda_2^3(W(\lambda_1) - W(r)) + \lambda_1^3(-W(\lambda_2) + W(r)) \\ & +\lambda_1(3r(2+r)(W(\lambda_1) - W(\lambda_2)) - 2\lambda_2(3+2r)(W(\lambda_1) - W(r))) \\ & \left. +\lambda_2^2(W(\lambda_1) - 4W(\lambda_2) + 3W(r))\right] \\ = & \sigma H_{2,CC}(r, \lambda_1, \lambda_2). \end{aligned}$$

Finally, for $t > T_{b2}$ the partial sum processes converge to

$$\begin{aligned} T^{-1/2}S_t \Rightarrow & \sigma\left[W(r) - W(1) - \frac{(1+2\lambda_2 - 3r)(r-1)(W(1) - W(\lambda_2))}{(\lambda_2 - 1)^2}\right. \\ & \left. + \frac{6(\lambda_2 - r)(r-1)\left((\lambda_2 - 1)W(1) + \int_0^1 W(s)\,ds + (1-\lambda_2)W(\lambda_2)\right)}{(\lambda_2 - 1)^3}\right]. \end{aligned}$$

Notice that the numerator of the KPSS test can be expressed as $T^{-2}\sum_{t=1}^{T} S_t^2 = T^{-1}\sum_{t=1}^{T}\left(T^{-1/2}S_t\right)^2 = T^{-1}\sum_{t=1}^{T_{b1}}\left(T^{-1/2}S_t\right)^2 + T^{-1}\sum_{t=T_{b1}+1}^{T_{b2}}\left(T^{-1/2}S_t\right)^2 + T^{-1}\sum_{t=T_{b2}+1}^{T}\left(T^{-1/2}S_t\right)^2$, which is shown to converge to

$$\begin{aligned} T^{-2}\sum_{t=1}^{T} S_t^2 \Rightarrow & \sigma^2\left[\int_0^{\lambda_1}(H_{1,CC}(r,\lambda_1,\lambda_2))^2\,dr + \int_{\lambda_1}^{\lambda_2}(H_{2,CC}(r,\lambda_1,\lambda_2))^2\,dr\right. \\ & \left. + \int_{\lambda_2}^{1}(H_{3,CC}(r,\lambda_1,\lambda_2))^2\,dr\right]. \end{aligned}$$

Following Phillips (1987) and Phillips and Perron (1988), Kwiatkowski, Phillips, Schmidt, and Shin (1992) use (4) as a consistent estimator for the error term variance so that the test statistic has as a limit distribution the following expression

$$\hat{\eta}_{CC} \Rightarrow \left[\int_0^{\lambda_1} \left(H_{1,CC}\left(r,\lambda_1,\lambda_2\right)\right)^2 dr + \int_{\lambda_1}^{\lambda_2} \left(H_{2,CC}\left(r,\lambda_1,\lambda_2\right)\right)^2 dr + \int_{\lambda_2}^{1} \left(H_{3,CC}\left(r,\lambda_1,\lambda_2\right)\right)^2 dr \right].$$

Therefore, Theorem 1 has been proved. ■

6.2 Consistency

As before, we focus on the Model CC since it is the most general specification that have been considered in the paper. Similar developments can be carried out for the other models.

The proof of the consistency of the tests is based on the divergence of the test statistic under the alternative. Let us first concentrate in the $T^{-2}\sum_{t=1}^{T} S_t^2$ factor of the KPSS with two structural breaks under the alternative of unit root. Note that under the alternative r_t is an I(1) process so that terms involving this variable will dominate the asymptotic distribution of the numerator. Estimated residuals $\hat{e}_t = y_t - \hat{y}_t = r_t - z_t\left(z'z\right)^{-1} z'r + \varepsilon_t - z_t\left(z'z\right)^{-1} z'\varepsilon$, define the partial sum processes given by $T^{-3/2}S_t = T^{-3/2}\sum_{i=1}^{[bT]} r_i - T^{-3/2}\sum_{i=1}^{[bT]} z_i P\left(Pz'zP\right)^{-1} Pz'r + T^{-3/2}\sum_{i=1}^{[bT]} \varepsilon_i - T^{-3/2}\sum_{i=1}^{[bT]} z_i P\left(Pz'zP\right)^{-1} Pz'\varepsilon$.

The third and fourth moments in the right hand are $o_p(1)$ so we only concentrate on the first and second terms to compute the asymptotic distribution of the numerator. It is straightforward to see that $T^{-3/2}\sum_{i=1}^{[bT]} r_i \Rightarrow \omega \int_0^b W(s)\,ds$, since $t = [bT]$, $b \in [0,1]$, with $\omega^2 = \lim_{T\to\infty} E\left(T^{-1}S_T^2\right)$, $S_T = \sum_{j=1}^{T} u_j$, and $[bT]$ is the integer part of bT. Regarding the second summand, it is necessary to distinguish between the three subperiods defined by the two breaks. Therefore, for $0 < b \leq \lambda_1$

$$T^{-1/2}\sum_{i=1}^{[bT]} z_i P \Rightarrow \omega \begin{bmatrix} b & \frac{b^2}{2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

for $\lambda_1 < b \leq \lambda_2$

$$T^{-1/2}\sum_{i=1}^{[bT]} z_i P \Rightarrow \omega \begin{bmatrix} b & \frac{b^2}{2} & (b-\lambda_1) & \frac{(b-\lambda_1)^2}{2} & 0 & 0 \end{bmatrix},$$

and for $\lambda_2 < b < 1$

$$T^{-1/2}\sum_{i=1}^{[bT]} z_i P \Rightarrow \omega \begin{bmatrix} b & \frac{b^2}{2} & (b-\lambda_1) & \frac{(b-\lambda_1)^2}{2} & (b-\lambda_2) & \frac{(b-\lambda_2)^2}{2} \end{bmatrix}.$$

The previous proof has established that $(Pz'zP) \to A$, so that we have to concentrate on the limit of the part involving r_t. It is easy to see that $T^{-1}Pz'\varepsilon \Rightarrow C$, where $C[1,1] = \omega \int_0^1 W(s)\,ds$, $C[2,1] = \omega \int_0^1 sW(s)\,ds$, $C[3,1] = \omega \int_{\lambda_1}^1 W(s)\,ds$, $C[4,1] = \omega \int_{\lambda_1}^1 (s-\lambda_1) W(s)\,ds$, $C[5,1] = \omega \int_{\lambda_2}^1 W(s)\,ds$, $C[6,1] = \omega \int_{\lambda_2}^1 (s-\lambda_2) W(s)\,ds$. Then, for $0 < b \leq \lambda_1$, the partial sum processes converge to

$$\begin{aligned} T^{-3/2}S_t &\Rightarrow \omega \left[\int_0^b W(s)\,ds - \begin{bmatrix} b & \frac{b^2}{2} & 0 & 0 & 0 & 0 \end{bmatrix} A^{-1}C \right] \\ &= \omega K_{1,CC}, \end{aligned}$$

for $\lambda_1 < b \leq \lambda_2$ they converge to

$$\begin{aligned} T^{-3/2}S_t &\Rightarrow \omega \left[\int_0^b W(s)\,ds - \begin{bmatrix} b & \frac{b^2}{2} & (b-\lambda_1) & \frac{(b-\lambda_1)^2}{2} & 0 & 0 \end{bmatrix} A^{-1}C \right] \\ &= \omega K_{2,CC}, \end{aligned}$$

and, finally, for $\lambda_2 < b < 1$

$$\begin{aligned} T^{-3/2}S_t &\Rightarrow \omega \left[\int_0^b W(s)\,ds - \begin{bmatrix} b & \frac{b^2}{2} & (b-\lambda_1) & \frac{(b-\lambda_1)^2}{2} & (b-\lambda_2) & \frac{(b-\lambda_2)^2}{2} \end{bmatrix} A^{-1}C \right] \\ &= \omega K_{3,CC}. \end{aligned}$$

Therefore, $T^{-4} \sum_{t=1}^{T} S_t^2 \Rightarrow \omega^2 \left[\int_0^{\lambda_1} K_{1,CC}^2 db + \int_{\lambda_1}^{\lambda_2} K_{2,CC}^2 db + \int_{\lambda_2}^{1} K_{3,CC}^2 db \right]$, so that the numerator of the tests is $O_p(T^2)$ under the alternative. Regarding the denominator, the bound established for the long-run variance in Sul, Phillips, and Choi (2003) ensures that the estimated long-run variance is bounded above $T\tilde{\sigma}_{\psi}^2$, which implies that the denominator of the statistic is $O_p(T)$. Using all these elements we can see that the individual KPSS test statistic is of order $O_p(T)$ under the alternative hypothesis and diverges, so that the test is consistent. Then, KPSS test with two structural breaks diverge under the alternative and, then, its consistence has been proved. ∎

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Table 2: Asymptotic critical values for the test η_{AAn}

Sig.		$\lambda_1 \backslash \lambda_2$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
level		0.1	0.4758	0.3659	0.2802	0.2299	0.2275	0.2883	0.3664	0.4758
		0.2		0.3682	0.2832	0.2109	0.1835	0.2075	0.2897	0.3612
		0.3			0.2874	0.2077	0.1588	0.1620	0.2178	0.2937
		0.4				0.233	0.1733	0.1648	0.1811	0.2292
1%		0.5					0.2271	0.2108	0.2027	0.2239
		0.6						0.2919	0.2846	0.2862
		0.7							0.3666	0.3692
		0.8								0.4853
		0.1	0.3752	0.2957	0.2248	0.1878	0.1876	0.2309	0.2920	0.3749
		0.2		0.2862	0.2210	0.1725	0.1531	0.1682	0.2258	0.2957
		0.3			0.2277	0.1700	0.1343	0.1338	0.1720	0.2297
		0.4				0.1918	0.1471	0.1371	0.1510	0.1878
2.5%	λ_1	0.5					0.1864	0.1723	0.1678	0.1848
		0.6						0.2283	0.2252	0.2238
		0.7							0.2914	0.2896
		0.8								0.3786
		0.1	0.2992	0.2344	0.1839	0.1560	0.1581	0.1883	0.2339	0.3001
		0.2		0.2339	0.1802	0.1423	0.1289	0.1390	0.1821	0.2366
		0.3			0.1846	0.1401	0.1153	0.1165	0.1421	0.1859
		0.4				0.1585	0.1266	0.1165	0.1275	0.1571
5%	λ_1	0.5					0.1564	0.1443	0.1388	0.1547
		0.6						0.1834	0.1830	0.1847
		0.7							0.2328	0.2332
		0.8								0.3009
		0.1	0.2262	0.1789	0.1441	0.1258	0.1286	0.1455	0.1773	0.2260
		0.2		0.1775	0.1402	0.1148	0.1053	0.1116	0.1394	0.1791
		0.3			0.1432	0.1128	0.0959	0.0963	0.1136	0.1441
		0.4				0.1276	0.1049	0.0965	0.1043	0.1279
10%	λ_1	0.5					0.1267	0.1151	0.1120	0.1264
		0.6						0.1440	0.1409	0.1438
		0.7							0.1774	0.1785
		0.8								0.2289

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 3: Asymptotic critical values for the test η_{AA}

Sig.	$\lambda_1 \setminus \lambda_2$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
level	0.1	0.1456	0.1169	0.1247	0.1601	0.1616	0.127	0.1157	0.1444
	0.2		0.1192	0.1002	0.1219	0.1400	0.1175	0.1028	0.1214
	0.3			0.1252	0.1187	0.1253	0.1248	0.1147	0.1268
	0.4				0.1604	0.1405	0.1272	0.1399	0.1574
1%	0.5					0.1679	0.1191	0.1159	0.1590
	0.6						0.1223	0.1020	0.1260
	0.7							0.1205	0.1180
	0.8								0.1421
	0.1	0.1173	0.0973	0.1051	0.1293	0.1294	0.1070	0.0981	0.1179
	0.2		0.0979	0.0869	0.1012	0.1134	0.0983	0.0869	0.0998
	0.3			0.1042	0.0989	0.1048	0.1042	0.0966	0.1059
	0.4				0.1297	0.1158	0.1058	0.1137	0.1261
2.5%	0.5					0.1326	0.0981	0.0969	0.1285
	0.6						0.1028	0.0861	0.1057
	0.7							0.0997	0.0993
	0.8								0.1183
	0.1	0.0988	0.0834	0.0899	0.1073	0.1080	0.0910	0.0825	0.0992
	0.2		0.0843	0.0750	0.0860	0.0948	0.0849	0.0749	0.0845
	0.3			0.0895	0.0848	0.0898	0.0886	0.0831	0.0898
	0.4				0.1069	0.0958	0.0896	0.0934	0.1051
5%	0.5					0.1087	0.0833	0.0836	0.1061
	0.6						0.0894	0.0746	0.0900
	0.7							0.0856	0.0843
	0.8								0.0999
	0.1	0.0797	0.0699	0.0748	0.0858	0.0855	0.0750	0.0693	0.0801
	0.2		0.0701	0.0641	0.0710	0.0771	0.0702	0.0643	0.0696
	0.3			0.0739	0.0706	0.0740	0.0736	0.0691	0.0741
	0.4				0.086	0.0771	0.0742	0.0747	0.0850
10%	0.5					0.0859	0.0699	0.0698	0.0867
	0.6						0.0741	0.0633	0.0743
	0.7							0.0714	0.0699
	0.8								0.0815

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 4: Asymptotic critical values for the test η_{BB}

Sig.	$\lambda_1 \backslash \lambda_2$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
level	0.1	0.1524	0.1298	0.1098	0.104	0.1003	0.1175	0.1393	0.1565
	0.2		0.1205	0.1021	0.0892	0.0879	0.0963	0.1134	0.1357
	0.3			0.0966	0.0820	0.0780	0.0827	0.0974	0.1153
	0.4				0.0797	0.0751	0.0766	0.0888	0.1040
1%	0.5					0.0803	0.0829	0.0910	0.1032
	0.6						0.0968	0.0993	0.1089
	0.7							0.1200	0.1282
	0.8								0.1518
	0.1	0.1257	0.1076	0.0915	0.0849	0.0856	0.0966	0.1126	0.1294
	0.2		0.0986	0.0843	0.0752	0.0739	0.0798	0.0928	0.1127
	0.3			0.0807	0.0685	0.0656	0.0695	0.0802	0.0959
	0.4				0.0681	0.0637	0.0652	0.0740	0.0865
2.5%	0.5					0.0671	0.0686	0.0754	0.0852
	0.6						0.0803	0.0831	0.0909
	0.7							0.0981	0.1055
	0.8								0.1226
	0.1	0.106	0.0897	0.0766	0.0723	0.0738	0.0803	0.0937	0.1068
	0.2		0.0822	0.0715	0.0650	0.0634	0.0681	0.0785	0.0929
	0.3			0.0679	0.0601	0.0573	0.0599	0.0683	0.0802
	0.4				0.0591	0.0556	0.0565	0.0633	0.0737
5%	0.5					0.0590	0.0590	0.0646	0.0731
	0.6						0.0684	0.0709	0.0772
	0.7							0.0818	0.0886
	0.8								0.1021
	0.1	0.0848	0.0729	0.063	0.0603	0.0613	0.0664	0.0755	0.0864
	0.2		0.0669	0.0593	0.0540	0.0529	0.0562	0.0644	0.0754
	0.3			0.0561	0.0504	0.0484	0.0502	0.0568	0.0659
	0.4				0.0500	0.0473	0.0483	0.0528	0.0609
10%	0.5					0.0502	0.0500	0.0539	0.0606
	0.6						0.057	0.0584	0.0634
	0.7							0.0674	0.0722
	0.8								0.0831

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 5: Asymptotic critical values for the test η_{CC}

Sig. level	$\lambda_1 \backslash \lambda_2$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0.1	0.1439	0.1116	0.0852	0.0691	0.0704	0.0856	0.1098	0.1430
	0.2		0.1100	0.0835	0.0644	0.0556	0.0634	0.0849	0.1113
	0.3			0.0855	0.0652	0.0506	0.0503	0.0637	0.0845
	0.4				0.0699	0.0560	0.0501	0.055	0.0695
1%	0.5					0.0704	0.0629	0.0637	0.0707
	0.6						0.0874	0.0840	0.0858
	0.7							0.1122	0.1124
	0.8								0.1425
	0.1	0.1166	0.0917	0.0713	0.0596	0.0596	0.0713	0.0906	0.1157
	0.2		0.0900	0.0695	0.0545	0.0482	0.0541	0.0696	0.0901
	0.3			0.0705	0.0550	0.0438	0.0442	0.0545	0.0716
	0.4				0.0596	0.0483	0.0438	0.0479	0.0598
2.5%	0.5					0.0603	0.0537	0.0538	0.0594
	0.6						0.0727	0.0691	0.0723
	0.7							0.0927	0.0920
	0.8								0.1151
	0.1	0.0972	0.0772	0.0605	0.0518	0.052	0.0606	0.0765	0.0965
	0.2		0.0763	0.0591	0.0470	0.0424	0.0466	0.0586	0.0757
	0.3			0.0601	0.0474	0.0390	0.0389	0.0467	0.0605
	0.4				0.0518	0.0425	0.0389	0.0423	0.0518
5%	0.5					0.0521	0.0466	0.0470	0.0523
	0.6						0.0615	0.0586	0.0608
	0.7							0.0768	0.0766
	0.8								0.0966
	0.1	0.0788	0.0626	0.0508	0.0441	0.0444	0.0504	0.0625	0.0775
	0.2		0.0621	0.0487	0.0399	0.0364	0.0397	0.0485	0.0619
	0.3			0.0501	0.0398	0.0339	0.0340	0.0397	0.0500
	0.4				0.0442	0.0367	0.0340	0.0365	0.0442
10%	0.5					0.0444	0.0397	0.0399	0.0443
	0.6						0.0507	0.0486	0.0503
	0.7							0.0620	0.0625
	0.8								0.0780

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 6: Asymptotic critical values for the test η_{AB-BA}

Sig.	$\lambda_1 \backslash \lambda_2$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
level	0.1		0.1545	0.1304	0.1099	0.0985	0.1001	0.1072	0.1297	0.1507
	0.2	0.1438		0.1351	0.1148	0.0955	0.0866	0.0929	0.1047	0.1232
	0.3	0.1223	0.1139		0.1129	0.1132	0.1124	0.1128	0.1175	0.1249
	0.4	0.1372	0.1056	0.0921		0.0921	0.1090	0.1286	0.1485	0.1650
1%	0.5	0.1699	0.1338	0.0989	0.0820		0.0821	0.0993	0.135	0.1712
	0.6	0.1649	0.1491	0.1315	0.1075	0.0920		0.0926	0.1044	0.1348
	0.7	0.1259	0.1190	0.1142	0.1115	0.1141	0.1137		0.1125	0.1223
	0.8	0.1262	0.1044	0.0916	0.0889	0.0960	0.1145	0.1355		0.1433
	0.9	0.1529	0.1279	0.1089	0.101	0.0985	0.1083	0.1282	0.1553	
	0.1		0.1282	0.1071	0.0904	0.0821	0.0836	0.0900	0.1061	0.1228
	0.2	0.1166		0.1113	0.0939	0.0809	0.0742	0.0786	0.0883	0.1028
	0.3	0.1039	0.0946		0.0928	0.0939	0.0933	0.0933	0.0983	0.1058
	0.4	0.1132	0.0890	0.0781		0.0775	0.0892	0.1049	0.1217	0.1342
2.5%	0.5	0.1369	0.1097	0.0833	0.0710		0.0712	0.0840	0.1116	0.1367
	0.6	0.1356	0.1199	0.1064	0.0894	0.0786		0.0785	0.0888	0.1124
	0.7	0.1068	0.0981	0.0945	0.0935	0.0931	0.0941		0.0950	0.1011
	0.8	0.1037	0.0871	0.0783	0.0753	0.0817	0.0955	0.1119		0.1167
	0.9	0.1252	0.1045	0.0911	0.0841	0.0828	0.0893	0.1059	0.1295	
	0.1		0.1070	0.0897	0.0774	0.0701	0.0706	0.0765	0.0886	0.1032
	0.2	0.0973		0.0942	0.0803	0.0702	0.0649	0.0681	0.0760	0.0868
	0.3	0.0881	0.0798		0.0794	0.0790	0.0786	0.0788	0.0841	0.0914
	0.4	0.0961	0.0773	0.0670		0.067	0.0747	0.0878	0.1002	0.1107
5%	0.5	0.1126	0.0924	0.0718	0.0620		0.0621	0.0724	0.0918	0.1139
	0.6	0.1113	0.0993	0.0874	0.0752	0.0673		0.0674	0.0762	0.0955
	0.7	0.0924	0.0838	0.0798	0.0781	0.0792	0.0796		0.0800	0.0858
	0.8	0.0880	0.0752	0.0681	0.0653	0.0707	0.0819	0.0934		0.0980
	0.9	0.1043	0.0872	0.0776	0.0711	0.0708	0.0750	0.0896	0.1069	
	0.1		0.0873	0.0729	0.0636	0.0583	0.0589	0.0638	0.0721	0.0832
	0.2	0.0799		0.0769	0.0666	0.0590	0.0554	0.0575	0.0635	0.0724
	0.3	0.0726	0.0659		0.0652	0.0656	0.0642	0.0653	0.0696	0.0760
	0.4	0.0790	0.0647	0.0562		0.0559	0.0619	0.0702	0.0794	0.0881
10%	0.5	0.0889	0.0733	0.0600	0.0532		0.0532	0.0606	0.074	0.0893
	0.6	0.0876	0.0795	0.0697	0.0615	0.0567		0.0569	0.064	0.0786
	0.7	0.0766	0.0688	0.0648	0.0646	0.0650	0.0652		0.0664	0.0713
	0.8	0.0733	0.0634	0.0573	0.0555	0.0596	0.0675	0.0770		0.0801
	0.9	0.0840	0.0711	0.0637	0.0590	0.0592	0.0629	0.0731	0.0858	

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 7: Asymptotic critical values for the test η_{AC-CA}

Sig	$\lambda_1 \backslash \lambda_2$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
level	0.1		0.1401	0.1106	0.0870	0.0723	0.0711	0.0871	0.1087	0.1419
	0.2	0.1421		0.1144	0.0895	0.0747	0.0696	0.0739	0.0918	0.1177
	0.3	0.1213	0.1137		0.0889	0.0763	0.0885	0.1004	0.1093	0.1256
	0.4	0.1233	0.0926	0.0860		0.0718	0.0693	0.0965	0.1305	0.1666
1%	0.5	0.1634	0.1092	0.0752	0.0699		0.0698	0.0753	0.1111	0.1616
	0.6	0.1638	0.1342	0.1001	0.0704	0.0716		0.0862	0.0944	0.1258
	0.7	0.1225	0.1087	0.0972	0.0880	0.0769	0.0870		0.112	0.1172
	0.8	0.1183	0.0923	0.0757	0.0688	0.0754	0.0883	0.1110		0.1409
	0.9	0.1433	0.1090	0.0866	0.0694	0.0698	0.0875	0.1142	0.1436	
	0.1		0.1165	0.0912	0.0719	0.0612	0.0609	0.0729	0.0910	0.1154
	0.2	0.1162		0.0924	0.0749	0.0644	0.0606	0.0643	0.0770	0.0994
	0.3	0.0996	0.0928		0.0739	0.0648	0.0732	0.0827	0.0917	0.1049
	0.4	0.1028	0.0778	0.072		0.0606	0.0606	0.0802	0.1051	0.1311
2.5%	0.5	0.1281	0.0898	0.0650	0.0599		0.0602	0.0651	0.0909	0.1280
	0.6	0.1287	0.1070	0.0812	0.0602	0.0607		0.0728	0.0788	0.1040
	0.7	0.1037	0.0902	0.0799	0.0722	0.0653	0.0719		0.0906	0.0975
	0.8	0.0984	0.0783	0.0645	0.0599	0.0642	0.0738	0.0911		0.1155
	0.9	0.1152	0.0903	0.0715	0.0598	0.0604	0.0731	0.0937	0.1161	
	0.1		0.0977	0.0774	0.0619	0.0534	0.0529	0.0622	0.0770	0.0971
	0.2	0.0969		0.0779	0.0644	0.0560	0.0526	0.0562	0.0671	0.0840
	0.3	0.0845	0.077		0.0625	0.0566	0.0618	0.0685	0.0784	0.0891
	0.4	0.0879	0.0669	0.0621		0.053	0.0529	0.0676	0.0869	0.1064
5%	0.5	0.1075	0.0765	0.0568	0.0527		0.0529	0.0571	0.0768	0.1062
	0.6	0.1061	0.0875	0.0683	0.0528	0.0532		0.0619	0.0678	0.0889
	0.7	0.0882	0.0762	0.0681	0.0622	0.0567	0.0618		0.0766	0.0832
	0.8	0.0837	0.0674	0.0567	0.0529	0.0557	0.0634	0.0770		0.0973
	0.9	0.0966	0.0767	0.061	0.0523	0.0533	0.0626	0.0786	0.0976	
	0.1		0.0789	0.0629	0.0517	0.0459	0.0453	0.0514	0.0627	0.0796
	0.2	0.0788		0.0637	0.0539	0.0478	0.0456	0.0481	0.0563	0.0696
	0.3	0.0696	0.0632		0.0518	0.0481	0.0514	0.0556	0.0639	0.0733
	0.4	0.0727	0.0561	0.0517		0.0452	0.0454	0.0553	0.0697	0.0849
10%	0.5	0.0842	0.0636	0.0480	0.0453		0.0451	0.0487	0.0628	0.0843
	0.6	0.0840	0.0698	0.0558	0.0455	0.0457		0.0516	0.0565	0.0728
	0.7	0.0731	0.0626	0.0559	0.0513	0.0481	0.0518		0.0628	0.0684
	0.8	0.0693	0.0567	0.0486	0.0455	0.0478	0.0534	0.0638		0.0791
	0.9	0.0784	0.0633	0.0514	0.0449	0.0454	0.0520	0.0642	0.0796	

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 8: Asymptotic critical values for the test η_{BC-CB}

Sig.	$\lambda_1 \backslash \lambda_2$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
level	0.1		0.1420	0.1136	0.0888	0.0721	0.0739	0.0905	0.1192	0.1516
	0.2	0.1483		0.1119	0.0849	0.0662	0.0630	0.0761	0.0982	0.1266
	0.3	0.1238	0.1147		0.0861	0.0677	0.0593	0.0668	0.0858	0.1083
	0.4	0.1091	0.0979	0.0925		0.0723	0.0642	0.0649	0.0778	0.0995
1%	0.5	0.0979	0.0840	0.0760	0.0742		0.0737	0.0734	0.0818	0.0982
	0.6	0.0950	0.0778	0.0669	0.0644	0.0725		0.0928	0.1002	0.1099
	0.7	0.1075	0.0827	0.0656	0.0590	0.0664	0.0873		0.1191	0.1297
	0.8	0.1282	0.0983	0.0770	0.0639	0.0676	0.0855	0.1088		0.1480
	0.9	0.1501	0.1167	0.0925	0.0747	0.0719	0.0845	0.1128	0.1379	
	0.1		0.1148	0.0916	0.0732	0.0621	0.0630	0.0761	0.0978	0.1230
	0.2	0.1212		0.0919	0.0708	0.0575	0.0541	0.0634	0.0802	0.1047
	0.3	0.1025	0.0957		0.0728	0.0577	0.0513	0.0566	0.0706	0.0895
	0.4	0.0891	0.082	0.0765		0.0626	0.0549	0.0554	0.0659	0.0819
2.5%	0.5	0.0816	0.0700	0.0641	0.0630		0.0643	0.0619	0.0690	0.0811
	0.6	0.0801	0.0661	0.0570	0.0556	0.0619		0.0765	0.0829	0.0911
	0.7	0.0897	0.0689	0.0558	0.0506	0.0569	0.0720		0.0984	0.1045
	0.8	0.1045	0.0808	0.0638	0.0549	0.0573	0.0701	0.0912		0.1212
	0.9	0.1219	0.0956	0.0764	0.0629	0.0620	0.0711	0.0916	0.1137	
	0.1		0.0959	0.0767	0.0618	0.0541	0.0553	0.0657	0.0811	0.1022
	0.2	0.1018		0.0774	0.0608	0.0500	0.0471	0.0543	0.0678	0.0874
	0.3	0.0861	0.0811		0.0622	0.0499	0.0453	0.0491	0.0598	0.0755
	0.4	0.0754	0.0689	0.0657		0.0544	0.0479	0.0479	0.0561	0.0690
5%	0.5	0.0690	0.0597	0.0545	0.0547		0.0554	0.0535	0.0599	0.0699
	0.6	0.0686	0.0566	0.0492	0.0484	0.0544		0.0647	0.0689	0.0764
	0.7	0.0754	0.0597	0.0487	0.0452	0.0501	0.0619		0.0815	0.0872
	0.8	0.0871	0.0681	0.0548	0.0482	0.0500	0.0595	0.0755		0.1019
	0.9	0.1016	0.0801	0.0651	0.0545	0.0549	0.0609	0.0765	0.0960	
	0.1		0.0777	0.0633	0.0517	0.0463	0.0470	0.0540	0.0662	0.0825
	0.2	0.0829		0.0626	0.0502	0.0429	0.0407	0.0455	0.0558	0.0706
	0.3	0.0708	0.0661		0.0521	0.0427	0.039	0.0416	0.0494	0.0622
	0.4	0.0621	0.0564	0.0542		0.0463	0.0412	0.0411	0.0471	0.0577
10%	0.5	0.0575	0.0495	0.0459	0.0471		0.047	0.0453	0.0494	0.0576
	0.6	0.0574	0.0473	0.0415	0.0415	0.0463		0.0537	0.0564	0.0627
	0.7	0.0619	0.0497	0.0414	0.0392	0.0424	0.0516		0.0669	0.0705
	0.8	0.0710	0.0555	0.0458	0.0413	0.0428	0.0495	0.0620		0.0826
	0.9	0.0825	0.0657	0.0539	0.0466	0.0465	0.0511	0.0619	0.0780	

The critical values were obtained using 20,000 replications and 2,000 steps to approximate the Wiener processes.

Table 9: Empirical size and power of the test statistic

$\vartheta_i = (\mu, \beta, \theta_1, \gamma_1, \theta_2, \gamma_2), i = 1, \ldots, 4$

	$\vartheta_1 = (0,0,0,0,0,0)$			$\vartheta_2 = (1, 0.2, -5, -0.2, 2, 0.5)$		
$T \backslash \rho$	0.8	0.9	1	0.8	0.9	1
100	0.031	0.042	0.035	0.024	0.080	0.167
200	0.027	0.026	0.070	0.041	0.137	0.496
500	0.040	0.022	0.401	0.054	0.065	0.905

	$\vartheta_3 = (1, 0.2, 0, 0, -5, -0.2)$			$\vartheta_4 = (1, 0.2, 0, 0, 0, 0)$		
$T \backslash \rho$	0.8	0.9	1	0.8	0.9	1
100	0.043	0.057	0.098	0.037	0.041	0.049
200	0.039	0.062	0.273	0.029	0.031	0.075
500	0.038	0.044	0.713	0.050	0.029	0.401

Simulation experiment has specified the relative position of the structural breaks at $(\lambda_1, \lambda_2) = (0.2, 0.6)$. The long-run variance has been estimated using the Quadratic Spectral kernel with the bandwidth estimated as in Sul, Phillips and Choi (2003). $n = 2,000$ replications are carried out.