A Chance - Constrained Maximum Capture Location Problem

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Abstract

The paper presents a new model based on the basic Maximum Capture model, MAXCAP. The Chance – Constrained Maximum Capture model introduced a stochastic threshold constraint, which recognised the fact that a facility can be open only if a minimum level of demand is captured. A metaheuristic based on the MAX – MIN ANT search procedure together with TABU.-based algorithm is presented to solve the model. This is the first time that the MAX – MIN ANT system is adapted to solve a location problem. Computational experience and an application to a 55-node network are also presented.

1. INTRODUCTION

Up till now, the concept of market threshold has not been used so much in facility location decision models. The threshold concept is particularly relevant to retail location, as it is widely recognized in the retail literature that states “there is a minimum size of a market below which a place will be unable to supply a central good … and is here termed the threshold sales level for the provision of that good from the center” (Berry and Garrison, 1958, p.111, as cited by Shonkwiler and Harris, 1996). In this paper, a new model based on the basic Maximum Capture Model (MAXCAP) formulated by ReVelle (1986) is presented. The MAXCAP model seeks the location of a given number of facilities in a discrete network so as to maximize market share captured. The Chance – Constrained Maximum Capture model presented in this paper introduces two important modifications that makes the model much more realistic:

- The capture is not just based on proximity. Instead, the capture is determined by the gravity model proposed by HUFF (1964)[1].
- Secondly, and new, stochastic threshold constraint is introduced. A facility can be open if the probability that total demand assigned to that outlet is above the threshold level, is at least a desired probability.

The paper is organized as follows. Section 2 reviews the literature. Section 3 presents the chance – constrained maximum capture model. In Section 4, we developed the metaheuristic

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[1] Accordingly to this model, each facility has a known attractiveness level, and the probability that a customer selects a facility is proportional to its attractiveness and inversely proportional to some power of the distance to it.

to solve the problem. Section 5 presents some computational experience on different sized network. In section 6 an example is presented on a 55-node network. Finally, the conclusions are set out in Section 7.

2. LITERATURE REVIEW

Competitive Location Literature addresses the issue of optimally locating firms that compete for clients in space. The first study of this line is due to Hotelling (1929), where consumers were assumed to patronize the closest facility. Different models based on this assumption of consumer behaviour have been developed. The most relevant ones are based on Voronoi Diagrams and Location-Allocation models that jointly determine the optimal location of service facilities and the allocation of service areas to them (Hodgson (1978)).

Several lines of work have been developed in this field. The key one for this paper was developed by ReVelle (1986). ReVelle and his followers have constructed a group of models that examined competition among retail stores in a spatial market. The basic model was the Maximum Capture Problem (MAXCAP, ReVelle (1986)). This model selects the location of servers for an entering firm which wishes to maximizes its market share; the market is one in which competitor servers are already in position. This model has been adapted to different situations. The first modification introduced facilities that are hierarchical in nature and where there is competition at each level of the hierarchy (Serra, et. al. (1992)). A second extension took into account the possible reaction from competitors to the entering firm (Serra and ReVelle (1994)). Finally, another modification of the MAXCAP problem introduced scenarios with different demands and / or competitor locations (Serra and ReVelle (1996)). A good review of these models can be found in Serra and ReVelle (1996).

All these Competitive Location theories find optimal locations assuming that customers patronize the closest shop. **Store – Choice literature** studies the key variables that influences a consumer when deciding where shop as well as the interaction between these variables. A good review can be found in Craig, et. al. (1984). Literature on the subject reveals that distance is not the only variable consumers take into account when deciding where to make their purchase. Eiselt and Laporte (1989) and Santos – Peñate, et. al. (1996) have introduced these concepts in the basic MAXCAP model. Recently, Colomé and Serra (2000) present an empirical study of the methodology to choose and introduce the key store choice attributes in the MAXCAP model.

Another assumption used in the basic MAXCAP model is the posibility to locate an outlet, regardless the level of demand capture. Several authors have recognised that there is a **demand entry threshold**. They have introduced this concept in the facility location decision models in different ways.

Balakrishnan and Storbeck (1991) presents the McTHRESH model. This model addressed the issue of locating a given number of outlets so that market coverage was maximised within some predetermined range and the required threshold level of demands were maintained for all sites. In 1994, Current and Storbeck (1994) formulated a multiobjective model that selected franchise locations and identified individual franchise market areas. Constraints in their formulation guarantee that all francise locations were assigned at least a minimal threshold market area with sufficient demand to ensure economic survival.

Recently, Serra, ReVelle and Rosing (1999) presented a decision model for a firm that wished to enter a competitive market where several competitors were already located. The market was such that for each outlet there was a demand threshold level that had to be achieved in order to survive. In this model, the threshold constraint was deterministic and each facility must meet the threshold.

Finally, Drezner and Drezner (2001) presented a location model based on the threshold concept. They assumed that the buying power at each community over the planning horizon was distributed according to some statistical distribution. Assuming that there was a minimum market share threshold to be captured, they introduced the threshold in the objective function. Their location objective become the minimisation of the probability of falling short of the required threshold.

In this paper, we present a model with a stochastic threshold, but as a constraint.

3. THE MODEL

The basic states that a new firm (from now on Firm A) wants to enter with p facilities in a market in order to obtain the maximum capture, given that it has to compete with q existing outlets[2], and subject to a threshold constraint that is stochastic. The demand of each node a_i is normally distributed with mean μ_i and standard deviation σ_i.

This model studies the location of retail facilities in discrete space. The model take the following assumptions:

- Demand is totally inelastic.
- The good is homogeneous.

[2] These competitors can belong to one or more firms, but without loss of generality it is assumed that there is only one competing firm (Firm B) operating in the market; as was assumed by ReVelle (1986).

- The threshold level is defined as the minimum expected amount of demand necessary to cover costs or as the minimum number of customers required[3].

- Price is set exogenously and consumers bear transportation costs.

- Under equal conditions, the existing firm captures the demand, following Hakimi assumption (1986).

- The demand of each node is drawn from a multivariate normal distribution. Note that according to the central limit theorem, it is not essential for these distribution to be normal if there are more that 30 nodes.

- The distributions of demand of two nodes are positively correlated or either uncorrelated (Drezner & Drezner, 2001).

- We use the simple **gravity model** to define the capture. According to these models, "the probability that a consumer patronises a shop (or the proportion of demand capture form a node by one shop) is proportional to its attractiveness and inversely proportional to a power of distance to it" (Reilly, 1929). In this paper, we used the simple HUFF model[4] (Huff, 1964).

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[3] Demand thresholds are usually measured in terms of population required to support one firm (Shonkwiler and Harris, 1996).

[4] The Huff probability formulation uses distance (or travel time) from consumer's zones to retail centers and the size of retail centers as inputs to find the probability of consumers shopping at a given retail outlet. He was also the first one to introduce the Luce axiom of discrete choice[4] in the gravity model. Using this axiom, consumers may visit more than one store and the probability of visiting a particular store is equal to the ratio of the utility of that store to the sum of utilities of all stores considered by the consumers.

The integer programming formulation of the Chance – Constrained Maximum Capture Location problem is as follows:

MAX $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \ \rho_{ij} \ x_{ij}$

Subject to

(1) $\sum_{j=1}^{n} x_{ij} = p + q \quad , \ i = 1,...,m$

(2) $x_{ij} \le x_{jj} \quad , \ i = 1,\dots,m \ , j = 1,\dots, n$

(3) $P\left(\sum_{i=1}^{m} a_i \rho_{ij} x_{ij} \ge T \right) \ge \alpha \quad , j = 1,\dots, n$

(4) $\sum_{j \subset J} x_{jj} = p$

$x_{ij} = \{0,1\} \ \ x_{jj} = \{0,1\} \quad , \ i = 1,\dots,m \ , j = 1,\dots, n$

Where the parameters are:

$i, I =$ Index and set of consumers' zones or nodes (*1,…, m*).

$j, J =$ Index and set of potential locations for shops (*1,…,n*)..

$p =$ Number of facilities to locate

$d_{ij} =$ The network distances between consumers' zone *i* and a shop in *j*.

ρ_{ij} = The probability that consumers at location *i* will shop at shop *j*. (i.e., The proportion of capture that a shop in *j* will achieve by consumers' zone *i*), based on HUFF model

$$\rho_{ij} = \frac{A_j / d_{ij}^{\beta}}{\sum_{j \in J_B} A_j / d_{ij}^{\beta} + \sum_{j \in J} A_j / d_{ij}^{\beta} * x_{jj}}$$

where A_j = The attractiveness of shop *j* (as in HUFF, the size of the shop)

$\beta =$ Distance decay parameter (as in HUFF, is equal to 2)

$T =$ Threshold demand level

α = Desired probability of satisfying the threshold level

$a_i =$ Demand at consumers' zone *i* (unknown).

μ_i = Mean of a_i

σ_i = Standard deviation of a_i

And the variables are defined as follows:

$x_{ij} = 1$, if consumers' zone *i* is assigned to node *j*; 0, otherwise.

$x_{jj} =$ 1, if a shop of firm's A is opened at node *j*; 0, otherwise.

The constraint set basically that: constraint set (1) states that every consumer zone makes *p* + *q* assignments to the *p* new and *q* existing outlets. But for a demand node *i* to be assigned to a facility at *j*, there has to be a facility open at *j*; this is achieved by constraint set (2). The third group of constraints allows a facility to open at *j* only if the probability that the total demand assigned to node *j* was above than the threshold level, is at least the desired probability of satisfying this required threshold level. Finally, constraint (4) sets the number of outlets to be opened by the entering firm.

The objective function defines the total capture that the entering firm can achieve with the sitting of its p servers.

A deepest analysis of the deterministic equivalent of constraint set (3) states that:

We do it for one constraint *j*

The mean of $\sum_{i=1}^{m} a_i \rho_{ij} x_{ij}$ is equal to $\sum_{i=1}^{m} \mu_i \rho_{ij} x_{ij}$ and the standard deviation S_j is $\left[\bar{z}_j^T \, \mathrm{V} \, \bar{z}_j \right]^{1/2}$

where,

$$V = \begin{bmatrix} \sigma_1^2 & \cdots & r_{1m}\sigma_1\sigma_m \\ \vdots & \ddots & \vdots \\ r_{m1}\sigma_m\sigma_1 & \cdots & \sigma_m^2 \end{bmatrix} \quad \text{and} \quad \bar{z}_j = \left(\rho_{1j} x_{1j}, \ \cdots \ , \rho_{mj} x_{mj} \right)$$

We have that $P\left(\sum_{i=1}^{m} a_i \rho_{ij} x_{ij} \geq T \right) \geq \alpha$ which is equivalent to $P\left(\sum_{i=1}^{m} a_i \rho_{ij} x_{ij} \leq T \right) \leq 1-\alpha$

$$P\left(\frac{\sum_{i=1}^{m} a_i \rho_{ij} x_{ij} - \sum_{i=1}^{m} \mu_i \rho_{ij} x_{ij}}{S_j} \leq \frac{T - \sum_{i=1}^{m} \mu_i \rho_{ij} x_{ij}}{S_j} \right) \leq 1-\alpha$$

$$P\left(Z \leq \frac{T - \sum_{i=1}^{m} \mu_i \rho_{ij} x_{ij}}{S_j} \right) \leq 1-\alpha$$

$$F\left(\frac{T-\sum_{i=1}^{m}\mu_i\rho_{ij}x_{ij}}{S_j}\right)\leq 1-\alpha$$

$$\frac{T-\sum_{i=1}^{m}\mu_i\rho_{ij}x_{ij}}{S_j}\leq F_Z^{-1}(1-\alpha)$$

$$\frac{T-\sum_{i=1}^{m}\mu_i\rho_{ij}x_{ij}}{S_j}\leq K_{1-\alpha}$$

$$\sum_{i=1}^{m}\mu_i\rho_{ij}x_{ij}\geq T-K_{1-\alpha}S_j$$

$$\sum_{i=1}^{m}\mu_i\rho_{ij}x_{ij}+K_{1-\alpha}S_j\geq T$$

Finally, we look at the covariance matrix S_j

- If a_i are completely dependent, then $V=\begin{bmatrix}\sigma_1^2 & \cdots & \sigma_1\sigma_m\\ \vdots & \ddots & \vdots\\ \sigma_m\sigma_1 & \cdots & \sigma_m^2\end{bmatrix}$, therefore

$\left[\bar{z}_j^T \text{ V } \bar{z}_j\right]=\left(\sum_{i=1}^{m}\sigma_i\rho_{ij}x_{ij}\right)^2$ and $S_j=\sum_{i=1}^{m}\sigma_i\rho_{ij}x_{ij}$. Then, the constraint is **not linear**

$$\sum_{i=1}^{m}\mu_i x_{ij}+K_{1-\alpha}\sum_{i=1}^{m}\sigma_i\rho_{ij}x_{ij}\geq T$$

- If a_i are completely independent, then $V=\begin{bmatrix}\sigma_1^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_m^2\end{bmatrix}$, therefore

$\left[\bar{z}_j^T \text{ V } \bar{z}_j\right]=\left(\sum_{i=1}^{m}\sigma_i^2\rho_{ij}^2x_{ij}^2\right)$ and then, the constraint becomes:

$$\sum_{i=1}^{m}\mu_i x_{ij} + K_{1-\alpha}\left(\sum_{i=1}^{m}\sigma_i^2\rho_{ij}^2 x_{ij}^2\right)^{\frac{1}{2}} \geq T$$ which **is not linear**.

Then, constraint set (3) in general is a non-linear constraint.

This non-linearity of the constraint set (3) don't allow to solve the model using the traditional methods of linear programming and branch and bound. Then, a metaheuristic model is used to solve it.

4. METAHEURISTIC TO SOLVE THE MODEL

The model presented in the previous section is a combinatorial optimisation problem. Many combinatorial problems are intractable and belong to the class of *NP*-Hard (non-deterministic polynomial-time complete) problems. Kariv and Hakimi (1979) prove that the p-Median problem is a NP-Hard problem on a general graph. Moreover, in this case, the inclusion of a non-linear constraint reinforce the NP-Hard condition of the problem.

The common belief in this field is that no efficient algorithm could ever be found to solve these inherently hard problems. Heuristics, and recently metaheuristics are considered one of the search methods for solving hard combination optimisation problems.

The basic families of metaheuristics are: genetic algorithms, greedy random adaptive search procedures, problem-space search, simulated annealing, tabu search, threshold algorithms and heuristic concentration (good review can be found in Osman (1995)).

Lourenço and Serra (2000) present new metaheuristics for the Generalised Assignment Problem. The best result was found using a MAX-MIN ANT SYSTEM + TABU SEARCH. In this paper, we are going to adapt this best metaheuristic to our location problem.

The Ant System

The Ant System introduced by Colorni, Dorigo and Maniezzo (1991a, 1991b), Dorigo et al. (1996), Dorigo and Di Caro (1999), is a cooperative search algorithm inspired by behavior of real ants. Ants lay down in some quantity an aromatic substance, known as pheromone, on their way to food. An ant chooses a specific path in correlation with the intensity of the pheromone. The pheromone trail evaporates over time if no more pheromone is laid down by other ants, therefore the best paths have more intensive pheromone and higher probability of being chosen. The Ant System approach associates pheromone trails to features of the solutions of a combinatorial problem, and can be seen as a kind of adaptive memory of the previous solutions. Solutions are iteratively constructed in a randomized heuristic fashion biased by the pheromone trails left by the previous ants. The pheromone trails, τ_j, are updated after the construction of a solution, ensuring that the best features will have a more intensive pheromone.

Recently, Stützle (1997) have proposed an improved version of the Ant System, designated by MAX-MIN Ant System. The MAX-MIN ant system differs from the Ant System in the following way: only the best ant updates the trails in every cycle. To avoid stagnation of the search, i.e. ants always choosing the same path, Stüzle (1998a) proposed a lower and upper limit to the pheromone trail, $\tau_{\min}$ and $\tau_{\max}$, respectively.

Stützle and Hoos(1999), Stützle (1997,1998a) applied this procedure to Traveling Salesman Problem, Quadratic Assignment Problem and Flow-Shop Scheduling Problem. Lourenço and Serra (2000) applied to the Generalised Assignment Problem.

Until now, the MAX-MIN ant system has not been applied to any location problem. In this paper, we propose a MAX-MIN Ant System with Local Search for the Chance – Constrainted Maximum Capture Location Model .

In our metaheuristic, we define τ_j as the desirability of locating a shop in j. Initially, $\tau_j = \sum_{i=1}^{m} \frac{A_j}{d_{ij}^{\lambda}}$. The more attractive the index of a shop in *j* is, the more desired is the location of an outlet in that node.

In the first step of the iteratively procedure of the MAX-MIN Ant System, a initial solution is constructed. To do this, the nodes are ordered with respect to the probability function defined by $p_j = \frac{\tau_j}{\sum_{l=1}^{n} \tau_l}$. The initial solution is choose randomly, taking into account the probability distribution defined previously.

The second step of the iteratively procedure tries to improve this initial solution by a local search method; specifically, Teitz and Bart heuristic. In both steps, only feasible solutions are allowed.

Finally, in the third step of the iteratively procedure, the pheromone trails are updated using the current solution, in the following way: $\tau_j^{new} = \rho \tau_j^{old} + \Delta \tau_j$, where ρ, 0 < ρ < 1, is the persistence of the trail, i.e. 1 - ρ, represents the evaporation.

The updated amount is $\Delta \tau_j = \quad \tau_{\max} * Q$ if an outlet is located in j; 0, otherwise. Finally, the MAX-MIN limits were imposed $\tau_{\min} \leq \tau_j \leq \tau_{\max}, \forall j$, if the updated pheromone falls outside the interval. The values of the parameters of the metaheuristic were set to Q = 0.05, ρ = 0.75, $\tau_{\max} = p * \max \tau_j$ and $\tau_{\min} = (1/p) * \min \tau_j$ (where, p is the number of outlets to locate).

The termination condition of this iteratively procedure is the number of total iterations (in this case, 30 iterations).

Tabu Search

In essence, **Tabu Search** (Glover, 1989,1990) explores a part of the solution space by repeatedly examining all neighbourhoods of the current solution, and moving to the best neighbourhood even if this deteriorates the objective function. This approach tries to avoid being trapped in a local optimum. In order to avoid the cycling solution that has recently been examined, nodes are inserted in a tabu list that is constantly updated. Additionally, several criteria of flexibility can be used in the tabu search including aspiration, intensification, diversification and stopping criteria.

This method has been successfully applied to a wide variety of location problems: p-hub location problems (Klincewicz (1992) and Marianov, et.al. (1997)), (r | *Xp)*- Medianoid and (r | *p*)- Centroid Problems (Benati and Laporte (1994)), the Vehicle Routing Problem (Gendreau, et.al. (1994)) and p-Median problem (Rolland, et.al. (1996)).

We have followed **BENATI and LAPORTE (1994)** application of Tabu Search. In their application, an aspiration criteria and a diversification criteria were applied

Summing up, the structure of the metaheuristic applied in this paper is the following:

<u>MAX-MIN ANT SYSTEM + TABU SEARCH</u>

MAX-MIN ANT SYSTEM

1. Initialise the pheromone trails and parameters.
2. While (termination condition is not met),

   2.1. Construct a solution using the Ant System Heuristic.

   In the first iteration initialise X_b, the best solution.

   2.2. Apply local search (x).

Here, the termination condition is the number of total iterations

2.3. Update the pheromone trails using the current solution x.

2.4. If x is feasible and $f(x) > f(x_b)$, let $x_b=x$.

3. Return the best solution found; x_b.

TABU SEARCH

3. Apply TABU SEARCH

3.1. Generate an initial solution x (in this case, take the best solution found in 2).

3.2. While the stopping criteria is not met do:

3.2.1. Generate the candidate list of moves / neighbourhoods.

3.2.2. Choose the best neighbour not tabu or verifying the aspiration criteria, x';

3.2.3. Update the current solution x = x'.

3.3. Output the best solution found.

5. COMPUTATIONAL EXPERIENCE

The algorithm has been applied to several randomly generated networks, having the number of nodes n equal to 35, 50 and 70. For each n, three different threshold level C were set using the following formula: $C = \beta\left[pop/(p+q)\right]$, where *pop* is the total amount of demand to be served, defined as $pop = \sum_i \mu_i$; and β is a threshold factor that was set to 0.1, 0.2 and 0.3[5].

For the threshold constraint, we assumed α = 95% ⇒ $z_{1-\alpha}$ = -1.645 (because, one-tailed test (left-tailed test) is applied).

We assume that there are five existing outlets. For each generated network, the location of the five existing outlets were found using the Teitz and Bart heuristic with a weighted total distance objective (i.e., minimised weighted by the population / demand of each node).

For each n, and each C, three different numbers of outlets of the entering firm were used; p = 2,3,4.

In this case, to generate the networks, the distributions of the demand nodes need to be established. In this case, we assume that the demand nodes follow a multivariate normal distribution $a_i \approx N(\mu_i, \sigma_i^2)$. This distribution will be established in the following way:

$\mu_i = uniform\ (50 - 100)$ and $\sigma_i{}^2 = \frac{\mu_i}{4}(uniform\ (0.2 - 0.8))$. We also give a priori value to the correlation between different demand nodes. This can be either unrelated or positively related (i.e., $r = 0$ or 0.1, as in Drezner and Drezner 2001)

Finally, we also need to pre-establish the value of the attractiveness of each shop ($A_i = uniform\ (60,100)$). It can be assumed that the attractiveness level represents the size of the shops .

Summing up, for each n, each β, each p and each r; ten networks were randomly generated. Therefore, a total of 540 networks were generated.

Optimal solutions were obtained using complete enumeration. The heuristic was programmed in FORTRAN and executed in Pentium III 450 Mhz with 128 mb of RAM. Results of heuristic performance are shown in table 1 and 2.

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[5] The computation of C is made a posteriori, when the distribution of the demand nodes is determined.

Table 1. Heuristic Performance (r = 0)

n	*β*	*(p,q)*	*Optimal Solutions*	*Lack of solution*	*Max-Min Average Deviation (%)*	*Total Average Deviation* (%)*
35	**0,1**	**2,5**	100%		5,10%	0,00%
		3,5	100%		6,91%	0,00%
		4,5	70%		8,52%	1,29%
	0,2	**2,5**	90%		4,57%	1,05%
		3,5	80%		8,13%	2,02%
		4,5	80%		7,82%	1,13%
	0,3	**2,5**	100%		2,26%	0,00%
		3,5	90%		6,30%	0,42%
		4,5	70%		9,16%	2,90%
50	**0,1**	**2,5**	100%		4,29%	0,00%
		3,5	100%		4,29%	0,00%
		4,5	70%		7,41%	2,15%
	0,2	**2,5**	90%		4,05%	1,70%
		3,5	80%		4,47%	1,46%
		4,5	70%		6,80%	2,04%
	0,3	**2,5**	100%		1,54%	0,00%
		3,5	90%		6,05%	1,40%
		4,5	70%		6,23%	1,06%
70	**0,1**	**2,5**	90%		1,50%	0,10%
		3,5	80%		1,71%	1,20%
		4,5	70%		4,99%	0,49%
	0,2	**2,5**	90%		1,01%	3,30%
		3,5	90%		1,58%	1,90%
		4,5	70%		5,90%	0,49%
	0,3	**2,5**	90%		1,28%	1,27%
		3,5	90%		2,59%	1,72%
		4,5	70%		3,65%	0,91%

* For non-optimal solutions

Table 2. Heuristic Performance (r = 0.1)

n	*β*	*(p,q)*	*Optimal Solutions*	*Lack of solution*	*Max-Min Average Deviation (%)*	*Total Average Deviation* (%)*
35	**0,1**	**2,5**	100%		5,41%	0%
		3,5	80%		5,67%	2,36%
		4,5	80%		8,56%	1,11%
	0,2	**2,5**	100%		4,85%	0%
		3,5	100%		8,27%	0%
		4,5	70%		5,87%	1,66%
	0,3	**2,5**	90%		4,58%	3,39%
		3,5	100%		6,34%	0%
		4,5	80%		8,92%	0,86%
50	**0,1**	**2,5**	100%		1,91%	0%
		3,5	90%		6,03%	0,42%
		4,5	70%		5,81%	1,24%
	0,2	**2,5**	90%		3,15%	0,78%
		3,5	80%		4,94%	3,81%
		4,5	70%		5,44%	2,23%
	0,3	**2,5**	100%	20%	21,94%	0%
		3,5	90%	30%	1,88%	0,36%
		4,5	80%	20%	23,36%	0,62%
70	**0,1**	**2,5**	100%		0,36%	0%
		3,5	80%	10%	2,89%	1,23%
		4,5	70%	20%	2,77%	2,31%
	0,2	**2,5**	100%	40%	0,21%	0%
		3,5	90%	70%	1,99%	0,73%
		4,5	100%	100%	0%	0%
	0,3	**2,5**	100%	100%	0%	0%
		3,5	100%	100%	0%	0%
		4,5	100%	100%	0%	0%

* For non-optimal solutions

The percentages of optimal solutions are presented in the column labelled “optimal solutions”. If at least a no optimal solution is found among the ten runs, the average deviation from optimality in both stages of the metaheuristic are presented at the two last columns. In this case, it can be noticed that the stochastic condition of the model arises the difficulty to find the optimal solutions. With the metaheuristic, a near-optimal solutions were found with a minimal deviation.

- **r = 0**. 41 out of 270 runs were non-optimal based on our comparison with complete enumeration. The maximum average deviation form optimality did not exceed 3.3 %.

- **r = 0,1**. 29 out of 270 runs were non-optimal based on our comparison with complete enumeration. The maximum average deviation form optimality did not exceed 3.9 %.

In both tables, an additional column has been included. The column “lack of solution” represents the percentages of cases without a solution; in other words, a network where the entering firm cannot find a solution that satisfied all the constraints; included the threshold constraint. It can be noticed that this lack of solution appears in table 2 with an r = 0.1. We can deduce, from previous models without the stochastic threshold constraint, that this constraint is the one no satisfy in these cases. A statistical interpretation of this fact states that a greater r implies a greater S_j , and as $K_{1-\alpha}$ is negative, the threshold constraint is more difficult to achieve. An economic interpretation of this output could be the following: correlation between demand nodes can be interpreted as that a higher demand power of one node implies a higher demand power of the others nodes. In this scenario, the established firm will capture more demand, by our initial assumption, and then, the entering firm will have more problems to find its outlets locations which satisfy the threshold constraint.

Tables 3 and 4 show the average execution time in seconds spend per phases by global metaheuristic and per enumeration procedure.

Table 3. Time Performance (r = 0)

n	*β*	*(p,q)*	*Max-Min Average Time*	*TABU Average Time*	*TOTAL Average Time*	*Enumeration Average Time*
35	**0,1**	**(2,5)**	0,20	1,34	1,55	0,44
		(3,5)	0,34	3,08	3,42	7,80
		(4,5)	0,36	5,51	5,87	86,58
	0,2	**(2,5)**	0,36	1,34	1,70	0,46
		(3,5)	0,32	3,11	3,43	7,78
		(4,5)	0,33	5,60	5,93	86,59
	0,3	**(2,5)**	0,19	1,33	1,52	0,46
		(3,5)	0,35	3,07	3,42	7,79
		(4,5)	0,38	5,61	5,99	86,60
50	**0,1**	**(2,5)**	1,14	2,31	3,45	1,62
		(3,5)	1,16	5,44	6,60	40,40
		(4,5)	1,42	10,00	11,41	659,86
	0,2	**(2,5)**	0,60	2,29	2,88	1,60
		(3,5)	0,91	5,38	6,29	40,39
		(4,5)	1,77	9,86	11,63	659,84
	0,3	**(2,5)**	0,74	2,30	3,04	1,61
		(3,5)	1,01	5,38	6,39	40,39
		(4,5)	1,08	9,70	10,77	659,78
70	**0,1**	**(2,5)**	4,02	3,96	7,98	5,41
		(3,5)	8,33	9,18	17,50	192,03
		(4,5)	8,52	16,65	25,17	4458,96
	0,2	**(2,5)**	2,66	3,93	6,59	5,40
		(3,5)	5,11	9,14	14,25	191,70
		(4,5)	5,03	16,41	21,44	4451,67
	0,3	**(2,5)**	3,44	3,96	7,40	5,40
		(3,5)	4,29	9,25	13,53	191,76
		(4,5)	11,82	16,88	28,70	4452,62

Table 4. Time Performance (r = 0.1)

n	*β*	*(p,q)*	*Max-Min Average Time*	*TABU Average Time*	*TOTAL Average Time*	*Enumeration Average Time*
35	**0,1**	**(2,5)**	0,23	1,33	1,55	0,46
		(3,5)	0,57	3,08	3,64	7,80
		(4,5)	0,37	5,62	5,99	86,58
	0,2	**(2,5)**	0,28	1,31	1,60	0,47
		(3,5)	0,32	3,13	3,45	7,80
		(4,5)	0,51	5,55	6,06	86,59
	0,3	**(2,5)**	0,22	1,32	1,55	0,45
		(3,5)	0,45	3,08	3,53	7,80
		(4,5)	0,57	5,64	6,21	86,64
50	**0,1**	**(2,5)**	1,66	2,31	3,97	1,61
		(3,5)	1,11	5,31	6,42	40,39
		(4,5)	1,50	9,83	11,33	660,78
	0,2	**(2,5)**	1,17	2,31	3,48	1,62
		(3,5)	1,74	5,16	6,90	40,39
		(4,5)	3,35	9,67	13,01	659,72
	0,3	**(2,5)**	2,42	2,16	4,58	1,61
		(3,5)	2,85	4,99	7,84	40,40
		(4,5)	4,16	8,97	13,13	659,98
70	**0,1**	**(2,5)**	5,70	3,76	9,46	5,39
		(3,5)	9,75	8,47	18,22	191,98
		(4,5)	14,12	16,56	30,68	4463,64
	0,2	**(2,5)**	8,96	3,53	12,49	5,40
		(3,5)	12,30	7,74	20,05	191,66
		(4,5)	18,12	14,02	32,13	4459,10
	0,3	**(2,5)**	8,12	3,40	11,52	5,42
		(3,5)	13,66	7,74	21,40	191,67
		(4,5)	17,87	13,98	31,85	4459,01

The average computing time of the heuristic is similar, maintaining the others parameters equal, for a network assuming r = 0 and r =0,1. The average computing time of the heuristic increased with the number of nodes and the number of outlets, as expected. Notice that the algorithm becomes very useful when we have to locate 3 or more entering outlets, regardless of the constraint level. In these cases, the time spent by the algorithm is less than the one for the enumeration procedure. For example, in n = 70, p = 4 and β =0.2, the time spent by the algorithm is 21.44 seconds while the enumeration procedure spent 4451.67 seconds to find the same solution.

6. AN EXAMPLE

The model was also tested in the well-known Swain's (1974) 55-node network, see Figure A1 of the appendix. The demand at each node follows a multivariate normal distribution, considering: $\mu_i =$ the original demand of the Swain's network indicated in Table A1 of the appendix, and $\sigma_i{}^2 = \frac{\mu_i}{4}(uniform\ (0.2 - 0.8))$. In this case, the total amount of demand to be captured is not always equal to 3.575.

We also need to pre-establish the value of the attractiveness of each shop. In this case, we assume that all the shops have the same attractiveness, $(A_i = 100)$, regardless of node and ownership.

The model was solved to optimality by using complete enumeration. As in the previous section, the location of the five existing outlets were found using the Tetiz and Bart heuristic with the weighted total distance objective.

For the example, different scenarios were examined; which varies with respect to the number of outlets to be located by Firm A (p = 2, 3 and 4), and to the threshold level C: $C = \beta\left[{pop}/{(p+q)}\right]$ (where pop is the total amount of demand to be served; i.e., $pop = \sum_i \mu_i$ and $\beta =$ 0.3, 0.5 and 0.7). Results are shown in table 5, 6 and 7.

The locations and percentage[6] of demand captured by Firm B are computed before and after the entering of Firm A locates its outlets (using as objective function the one of the Chance – Constrained Maximum Capture Location Problem). Firm's A optimal locations and its percentage of demand capture are also computed. Additionally, the following values are also computed for each scenario:

- *% Capture > T* = $\frac{(Capture\text{ - Threshold level})}{\text{Threshold level}} * 100$; the percentage of capture above the threshold level achieve by each Firm's A location.
- *% Constraint A.* = $\frac{\left(\sum_{i=1}^{m} \mu_i \rho_{ij} x_{ij} + K_{1-\alpha} S_j - Threshold\text{ Level}\right)}{Thershold\text{ Level}} * 100$; the percentage of threshold constraint accomplishment.

Note that the percentage of capture above the threshold level measure the accomplishment of the threshold constraint in the actual event. While the percentage of threshold constraint accomplishment measures this value in the general characteristics of the market.

[6] Note that the percentage of demand captured is computed instead the total amount because the scenario is stochastic; i.e. the total demand varies from one to other scenario. In this way, it is easy the comparison

Table 5. 55-nodes example (r = 0 and β=0,3)

β	(p,q)	Firm's B Location	Initial Capture	Final Capture	Firm's A Location	Capture	% Capture > T	% Constraint A.
0,3	**(2,5)**	17	16%	14%	4	18%	311%	240%
		41	22%	13%	5	12%	178%	228%
		38	13%	12%				
		31	18%	16%				
		5	30%	15%				
		Total Capture	***100%***	***70%***		***30%***		
	(3,5)	12	12%	10%	4	15%	316%	284%
		41	22%	12%	5	13%	243%	243%
		38	14%	11%	31	11%	210%	209%
		31	21%	13%				
		5	31%	15%				
		Total Capture	***100%***	***61%***		***39%***		
	(4,5)	17	13%	11%	3	11%	234%	245%
		25	25%	10%	4	14%	327%	266%
		38	12%	9%	5	11%	233%	238%
		3	25%	14%	25	8%	137%	150%
		5	25%	14%				
		Total Capture	***100%***	***57%***		***43%***		

between scenarios.

Table 6. 55-nodes example (r = 0 and β=0,5)

β	(p,q)	Firm's B Location	Initial Capture	Final Capture	Firm's A Location	Capture	% Capture > T	% Constraint A.
0,5	**(2,5)**	22	16%	13%	3	17%	136%	94%
		25	21%	14%	5	15%	109%	124%
		38	11%	10%				
		31	21%	14%				
		6	31%	18%				
		Total Capture	***100%***	***69%***		***31%***		
	(3,5)	16	16%	11%	3	11%	75%	91%
		41	25%	12%	4	14%	141%	110%
		23	15%	11%	5	14%	130%	100%
		3	18%	13%				
		2	25%	14%				
		Total Capture	***100%***	***61%***		***39%***		
	(4,5)	22	17%	11%	3	13%	127%	96%
		20	17%	10%	4	15%	160%	124%
		31	24%	11%	5	10%	69%	107%
		38	11%	9%	31	9%	64%	74%
		5	32%	13%				
		Total Capture	***100%***	***54%***		***46%***		

Table 7. 55-nodes example (r = 0 and β=0,7)

β	(p,q)	Firm's B Location	Initial Capture	Final Capture	Firm's A Location	Capture	% Capture > T	% Constraint A.
0,7	**(2,5)**	12	10%	9%	5	18%	79%	77%
		25	20%	14%	31	12%	24%	29%
		31	24%	15%				
		38	14%	12%				
		5	32%	20%				
		Total Capture	***100%***	***70%***		***30%***		
	(3,5)	22	19%	13%	4	16%	79%	62%
		25	21%	12%	5	11%	28%	51%
		43	11%	10%	31	11%	22%	25%
		31	20%	13%				
		5	28%	14%				
		Total Capture	***100%***	***62%***		***38%***		
	(4,5)	22	19%	12%	3	14%	83%	53%
		20	15%	10%	4	14%	82%	65%
		38	11%	9%	5	11%	44%	54%
		18	19%	10%	18	8%	4%	19%
		5	36%	14%				
		Total Capture	***100%***	***54%***		***46%***		

From the previous tables, we can point out the following:

- The percentage of total demand achieved by the entering firm is the same for a given number of outlets located, regardless threshold level. For example, when the entering firm locates 3 outlets, it captures the 39%, 39% and 38% of total demand, with β=0,3, 0,5 and 0,7 respectively.

- Obviously, the percentage of capture above the threshold level and the percentage of threshold constraint accomplishment achieve by each Firm's A, decrease with an increase of β value.

7. CONCLUSIONS

In this paper, a new location model have been presented to study the issue of minimum requirements to survive in a given spatial setting. The threshold requirement has been introduced as an stochastic constraint. A metaheuristic based on MAX – MIN Ant System and TABU system has been used to solve the new model. It is the first time that the MAX – MIN Ant system is adapted to solve a location problem.

The model is particularly relevant to private retail sector setting because it takes into account two real characteristics of the market. First of all, the capture is determined by a gravity model, which is a revealed preference model. And secondly, the model included a threshold constraint which reflex the fact that a facility cannot be open if the demand captured is below a threshold level.

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APPENDIX

Table A1. 55-Node Demand of Swain’s (1974) Network.

Node	Demand	Node	Demand	Node	Demand
1	120	20	77	39	47
2	114	21	76	40	44
3	110	22	74	41	43
4	108	23	72	42	42
5	105	24	70	43	41
6	103	25	69	44	40
7	100	26	69	45	39
8	94	27	64	46	37
9	91	28	63	47	35
10	90	29	62	48	34
11	88	30	61	49	33
12	87	31	60	50	33
13	87	32	58	51	32
14	85	33	57	52	26
15	83	34	55	53	25
16	82	35	54	54	24
17	80	36	53	55	21
18	79	37	51		
19	79	38	49		