Scientific Collaboration Networks: The role of Heterogeneity and Congestion*

Antoni Rubí Barceló†

September, 2007

Abstract

We propose a dynamic model to analyze the formation of scientific collaboration networks. In this model, individuals continuously make decisions concerning the continuation of existing collaboration links and the formation of new links to other researchers through a link formation game. Once the network has been constituted, ideas arrive from outside of the network to every node at a constant rate and agents are able to require the collaboration of one of the previously selected coauthors to increase the chances to publish articles based on those ideas. Agents are heterogeneous -they have different levels of productivity-, and they have a limited processing capability; so congestion can arise when a researcher receives a sufficiently high amount of collaboration requests. Consequently, the decision of whether to form a link must consider the trade off between the rewards (or costs) from collaborating with more (or less) productive agents and the costs (or rewards) from collaborating with more (or less) congested co-authors.

Focusing on the role of heterogeneity among agents' productivity and congestion problems derived from their limited processing capability we show how self-interested researchers can organize themselves forming the kind of scientific collaboration network topologies observed in reality.

JEL classification: Z13, D85

Keywords: network formation game, scientific collaboration.

---

*I am grateful to Antonio Cabrales for very helpful comments and suggestions. I have also benefitted from comments by Joan de Martí, David Grandadam and Tomás Lejarraga. Thanks also to other participants in *Augustin Cournot Doctoral Days 2007*, *Economics and Management of Networks 2007*, *Conference of the Association of Southern European Economic Theorists 2007* and *Simposio de Análisis Económico 2007*. Financial support from Fundación Ramón Areces is gratefully acknowledged. All errors are of course mine.

†Universitat Pompeu Fabra, 25-27 Ramon Trias Fargas st, 08005 Barcelona. Phone: 93-542-2575, e-mail: antoni.rubi@upf.edu

1 Introduction

Social networks underlie many economic and social activities to the point that certain outcomes cannot be understood without taking into account the specific network structure. Examples and references are numerous[1]. One of the environments in which the key role of a social network is more evident is academics. In scientific production, the association with a group of competent colleagues to exchange information is a strong advantage in order to discover errors, raise research questions, and discern the appropriate ways to solve a problem. This unquestionable significance of networks in understanding scientific activity is one of the reasons that explain the extensive empirical work on this field. Today, in the advent of the information and communication revolution, data on scientific articles and researchers is stored in electronic databases containing thousands of records. With the use of these databases, empirical studies are able to reproduce coauthorship networks (in these networks a link between two researchers exists whenever there exists an article coauthored by them). From there, they are able to represent and analyze the main statistics of the collaboration among authors.

Empirical research about coauthorship networks is large[2]. Newman (2004), Newman (2001a) and Newman (2001b) analyze the defining statistics of coauthorship networks in Biology, Physics and Mathematics. Laband and Tollison (2000) focus on the importance of informal collaboration relationships in the comparison between networks in Economics and Biology. Hudson (1996) studies the reasons of the increase in the number of coauthors per paper in Economics. But the empirical work that most clearly shows these patterns of collaboration is Goyal, Van der Leij and Moraga (2006) (GVM hereafter). This work describes a detailed image of the features of actual coauthorship networks[3].

In spite of the great variety of empirical studies, there is a lack of foundational theoretical models that analyze how individual decisions contribute to the formation of scientific collaboration networks. To the best of our knowledge, chapter 4 in Van der Leij (2006) is the only attempt to compensate this deficiency. This paper, proposes a model that differs from Van der Leij (2006) but shares the same objective.

1.1 Characteristics of co-authorship networks

Before introducing the model, let us describe some of the key features of scientific collaboration networks. A surprising characteristic is the small average distance (measured by the shortest path length) between pairs of nodes. This stylized fact of social networks is captured in the famous "six degrees of separation" of John Gaure's play[4]. Scientific collaboration networks are not an exception

[1] Calvó-Armengol and Jackson (2004) on learning about job openings through contacts or Kranton and Minehart (2001) on buyer-seller networks are only two examples

[2] Albert and Barabási (2002) offers a survey of empirical studies about any type of networks.

[3] Although this empirical work refers to the field of Economics, we will argue that the main characteristics of co-authorship networks apply to other fields.

[4] Stanley Milgram (1967) pioneered the study of path length through a clever experiment where people had to send a letter to another person who was not directly known to them. The diameters of a variety of networks were

to this phenomenon as GVM shows. The average distance in the Economics coauthorship network they analyzed was 9.47 with a total population of 33,027 nodes (*i.e.* researchers). This regularity extends to other fields. Newman (2004) shows that the average distances are 4.6 in Biology, 5.9 in Physics and 7.6 in Mathematics.

Another interesting feature of coauthorship networks refers to the degree-distribution of nodes which tends to exhibit "fat tails", *i.e.* there is a small part of the population accumulating a large proportion of links. In particular, GVM found that the 20% of most-linked authors in Economics account for about 60% of all the links. Newman (2004) shows that this phenomenon also extends to coauthorship networks in the fields of Biology, Physics and Mathematics. In each case, the distribution is fat tailed, with a small fraction of scientists having a very large number of collaborators.

GVM shows that the best-connected researchers collaborate extensively and most of their coauthors do not collaborate with each other. Moreover, the authors observe that these individuals are essential in maintaining the connectivity of the network. On the other hand, Newman (2004) found that most of the connections (64%) of an individual's shortest path to other researchers pass through the best-connected of their collaborators, and most of the remainders pass through the next-best connected.

These results lead GVM to conclude that: "the world of Economics is spanned by inter-linked stars" (an inter-linked star is a network in which some nodes connected among them accumulate a lot of links with other nodes who are not connected among themselves). Despite that there is no such conclusion referred to co-authorship networks in other fields, the similarity in the general results showed in Newman (2004) suggests a similar pattern in Biology, Physics and Mathematics. Moreover, GVM analyzes the evolution over the last thirty years and concludes that such a structure is stable over time.

1.2 Preview of the model and results

We show that the effects of two simple driving forces can explain the formation of scientific collaboration networks with an interlinked star topology. These two forces are caused by the heterogeneity among researchers and their limited processing capability. Specifically, we propose a dynamic model in which individuals periodically make decisions concerning the continuation of existing collaboration links and the formation of new links to other researchers through a link formation game. Once the network has been constituted, ideas arrive from outside of the network to every node at a constant rate and agents are able to require the collaboration of one of the previously selected coauthors to increase the chances to publish articles based on those ideas. As commented above, agents are heterogeneous –they have different levels of productivity–, and they have a limited processing capability; so congestion can arise when a researcher receives a sufficiently high amount of collaboration requests. Consequently, the decision of whether to form a link must consider the trade off between the rewards (or costs) from collaborating with more (or less) productive agents and the costs (or

measured. These include purely social networks, co-authorship networks, parts of the internet and parts of the world wide web. See Albert and Barabási (2002) for an illuminating account.

rewards) derived from more (or less) congested coauthors.

After defining the rules of the network formation game, the timing and procedure of the publication process, the payoff function and the equilibrium concept in section 2, we assume that the process reaches a Steady State and we characterize it. Then, we study which kind of network topologies can be sustained in Steady State. This analysis is divided into two parts: first, in section 3.1 we show several results that sharply narrow the set of potential equilibrium networks. In particular, these results show that the concentration of links towards the researchers with a higher productivity is a natural feature of equilibrium topologies in our model. Notice that this distribution of links generates a clearly unequal situation in which some agents do not receive ideas from others and where others receive numerous collaboration requests. Roughly speaking, we show that, in equilibrium, highly productive agents receive the minimum number of links necessary to exhaust their processing capability. Although the commented networks are highly unequal, we find the necessary conditions to reproduce this kind of equilibrium networks even for a highly homogeneous population of researchers.

In section 3.2 we go one step further in the simplification of the set of potential Steady State networks, and identify the conditions under which a single topology with the basic characteristics of actual scientific collaboration networks can be sustained as the unique equilibrium network of our model. Thus, our model naturally reproduces the scientific collaboration patterns observed in reality.

1.3 Literature Review

Theoretical models of social network formation can be classified into two groups. On one hand, there are the physics-based modeling of society. This approach treats agents as if they were just matter. That is, agents are non-strategic. This set has its origins in the random graph literature and has examples in sociology and recently in computer science and statistical physics. References of this kind of models are abundant[5] but we will focus on two of them. Jackson and Rogers (2006) proposes a nice, simple and general model of network formation. The authors combine random meeting and network-based meeting in a natural manner and analyze the relevance of these two forces in determining the formation of different kinds of networks (scientific collaboration structures are one of them). The second model we focus on is Arenas et al (2003). The authors present a stylized model of a problem-solving organization –whose internal communication structure is given by a network– that can suffer congestion. The authors develop a design problem to determine which kind of network architectures optimizes performance for any given problem arrival rate. Contrarily to our model, the network is fixed and players are non strategic.

The second classification of models involves strategic formation of networks and use game theoretic tools. That is, there is no exogenous prescription of how the network is formed but there is a definition of the rules of the game that agents have to play to form the network (see Jackson (2004) for a survey of this type of models). The model presented here belongs to this group of models.

[5]See Newman (2003) for a survey. Some examples are Watts (1999), Cooper and Frieze (2003) or Price (1976).

As introduced above, the work that more closely relates to our model is chapter 4 in Van der Leij (2006). This author also attempts to develop a theoretical model to explain the empirical regularities of research collaboration networks. In both models, heterogeneity across researchers plays a key role in explaining the results. Contrarily to our paper, Van der Leij constructs a static model in which the cost of link formation and the specific academic rewards scheme affect the equilibrium network topologies. Our model is dynamic and involves the possibility of congestion as the key factor (joint with agents' heterogeneity) for obtaining the results. Moreover, we do not require a minimum degree of heterogeneity among researchers (as Van der Leij (2006)) to reproduce the equilibrium networks observed in reality.

2 General setting

Let N be the set of nodes, interpreted as researchers, with $n = |N|$ and let i and j be typical members of this set. We assume that n is finite and arbitrarily large. Networks are modeled as directed graphs. A directed graph on N is an $N \times N$ matrix g where entry g_{ij} indicates whether a directed link exists from node i to node j; $g_{ij} = 1$ indicates the existence of such a directed link and $g_{ij} = 0$ indicates the absence of this directed link. Notice that we do not impose any specific value for g_{ii}; in particular, it is possible to have $g_{ii} = 1$ (see interpretation below). For any node $i \in N$, let $N_i(g) = \{j \in N : g_{ji} = 1\}$ be the set of players that have a link to i and $\eta_i(g) = |N_i(g)|$ denote the in-degree of i. On the other hand, let $M_i(g) = \{j \in N : g_{ij} = 1\}$ be the set of destinations of the links of i and $\mu_i(g) = |M_i(g)|$ denote the out-degree of i. Notice that $\eta_i(g)$ and $\mu_i(g)$ have to be natural numbers. We impose that $\mu_i(g) \geq 1$.[6]

Time is modeled continuously. However, for descriptive convenience, we split time in periods. The object of the agents in this model is to publish papers. This is their only source of payoffs. Specifically, a publication provides one unit of payoff, which is equally split among all its coauthors. A publication starts with an idea. Researchers receive ideas from outside the network at an independent positive rate ρ. These ideas are *open*, in the sense that they need to be processed to become a publication. Immediately after receiving these open ideas, agents send them to some previously selected destination. Agents can also choose to retain ideas. Here it is where the network plays its role, because a researcher i can only send her open ideas to some agent $j \in M_i(g)$ ($i \in M_i(g)$, i.e. $g_{ii} = 1$, means that agent i retains (part of) her own open ideas). We assume that all agents in $M_i(g)$ have the same probability of being selected as destination of a particular open idea obtained by i. The node chosen as destination is the researcher in charge of starting the publication process of this idea.

At any time, several open ideas may "wait" to be processed by certain node (as in a queue) because we assume that researchers have a limited processing capability. Specifically, there is an upper-bound in the number of ideas a node can process per period which we normalize to one[7]. Therefore, if a researcher receives a sufficiently high amount of collaboration requests (*i.e.* links),

[6] when $\nexists j \neq i$ such that $g_{ij} = 1$, then g_{ii} must be necessarily 1. When $\exists j \neq i$ such that $g_{ij} = 1$, g_{ii} can also be 1.
[7] Notice that this assumption is not restrictive at all since the length of the period is not specified.

queues will be formed. Given this possibility, agents are provided with a decision rule to select the open idea they will process from their stock. We will take the simplest rule, that is, all open ideas in a queue have the same probability of being selected. Researchers also have a limited storage capability. In particular, each agent forgets an open idea with probability q. For this reason, not all open ideas received by a node will be finally processed.

Once an open idea is chosen to be processed, two outcomes are possible: it is published or rejected forever. In this setting, a publication can have at most two coauthors: the researcher who initially gets the open idea from out of the network and the destination of this open idea (notice that these two nodes can coincide). When an open idea is processed, the probability of being published by the coauthors (author) depends on their (her) talent. Let h be the vector of talent endowments and h_i be the i-th element of this vector interpreted as the agent i's amount of talent. We assume that h_i is exogenous, randomly generated following the probabilities described by a continuous distribution function[8] and that $h_i > 0$ for all $i \in N$. Vector h is fixed throughout the game. The relationship between talents and publication probabilities is determined by $f(\cdot)$. This is a strictly increasing probability function, holding $f(0) = 0$. This implies that the higher it is the amount of talent of a researcher/node the higher it is the probability of publishing the ideas being processed. So, $f(h_i + h_j)$ is the probability of publishing a particular idea processed by i (or j) and previously sent by j (or i). Notice that h_i can also be interpreted as the agent i's productivity.

Therefore, agents are characterized by two defining features: the endogenous size of their queue of open ideas waiting to be processed and the exogenous amount of talent.

2.1 Network formation game and timing

As commented before, time is modeled continuously but described in periods. At the beginning of a period, collaboration links are configured through the following network-formation game: all players $i \in N$ simultaneously announce the direct and directed links they wish to have either as origin or as destination. Formally, $S_i = \{0,1\}^{2n-1}$ is i's set of pure strategies. Let

$$s_i = (s^i_{i1}, s^i_{i2}, ..., s^i_{ii}, ..., s^i_{in}, s^i_{1i}, ..., s^i_{i-1,i}, s^i_{i+1,i}, ..., s^i_{ni}) \in S_i.$$

Then, $s^i_{ij} = 1$ if and only if player i wants to set up a directed link from i to j (and thus $s^i_{ij} = 0$, otherwise). As commented before $s^i_{ii} = 1$ is possible. A link, which is assumed to be costless, from player i to player j is formed if and only if $s^i_{ij} s^j_{ij} = 1$. That is, we assume that mutual consent is needed to create a link. Let $S = S_1 \times ... \times S_n$. A pure strategy profile $s = (s_1, ..., s_n) \in S$ induces a directed network $g(s)$.

Once the new network is formed, any agent (say i) receives open ideas at a rate ρ and sends them to one selected destination. Simultaneously, node i selects and processes open ideas from her stock (if any) at a rate of one idea per period. At the end of each period, all stocked open ideas are forgotten with probability q. Just before the end of a period, the stock of open ideas of all nodes is updated.

[8] Notice that this implies that the probability of two agents having exactly the same amount of talent is zero.

2.2 Steady State analysis and payoff function

Suppose that the process reaches a Steady State. There are two defining properties of the Steady State: each researcher's stock of open ideas is constant and the network is stable.

Let o_i be the Steady State stock of open ideas waiting to be processed by node i. Under stationarity, the number of open ideas in stock behaves as a Markov process and the arrival and departure of ideas from and to each node i follow Poisson processes. Given that in Steady State all open ideas that arrive to a node eventually depart from it in finite time, we must observe that the arrival rate of open ideas must be equal to its departure rate. That is:

$$\rho \sum_{l \in N_i(g)} \frac{1}{\mu_l} = \begin{cases} 1 + q o_i & \text{, if } o_i \geq 1 \\ o_i(1+q) & \text{, otherwise} \end{cases} \quad \forall\, i \in N$$

The arrival rate of ideas to agent i is equal to the sum, over all nodes sending to i in g, of the expected number of ideas they receive from out of the network per instant of time (ρ) times the probability of sending ideas to i. The departure rate is the sum of the processing rate and the rate of open ideas that node i forgets. Notice that when the stock of open ideas is lower than one the processing capability of a node will be restricted. In such a case, only o_i open ideas can be processed per period (on average). A node i is said to be *congested* if the Steady Staten stock of open ideas o_i is higher than one.

From the last expression, we write the Steady State stock of open ideas of a node as:

$$o_i = \begin{cases} \frac{\rho(\sum_{l \in N_i(g)} \frac{1}{\mu_l}) - 1}{q} & \text{, if } \sum_{l \in N_i(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho} \\ \frac{\rho(\sum_{l \in N_i(g)} \frac{1}{\mu_l})}{1+q} & \text{, otherwise} \end{cases} \quad \forall\, i \in N \tag{1}$$

The stock of open ideas of a node is completely determined by the network structure (g) and also by q and ρ.

The other defining feature of the Steady State is network stability. Before defining the stability concept we introduce the payoff function. As commented above, researchers only obtain profits from the publication of ideas. For a given network structure g, the following expression defines the expected payoff agent i obtains per period when the stock of open ideas is constant for all agents[9]:

$$\Pi_i(g) = \Theta(i)[\sum_{l \in N_i(g) \backslash i} \frac{1}{\mu_l} c f(h_l + h_i) + g_{ii} \frac{1}{\mu_i} f(h_i)] + \frac{1}{\mu_i} \sum_{l \in M_i(g) \backslash i} \Theta(l) c f(h_l + h_i) \text{ with } c = \frac{1}{2} \tag{2}$$

$$\text{where } \Theta(i) = \begin{cases} \frac{1}{\sum_{k \in N_i(g)} \frac{1}{\mu_k}} & \text{, if } \sum_{k \in N_i(g)} \frac{1}{\mu_k} \geq \frac{q+1}{\rho} \\ \frac{\rho}{1+q} & \text{, if } 0 < \sum_{k \in N_i(g)} \frac{1}{\mu_k} < \frac{q+1}{\rho} \\ 0 & \text{, if } \sum_{k \in N_i(g)} \frac{1}{\mu_k} = 0 \end{cases}.$$

Agent i's publications can derive from her own stock of open ideas or from the open ideas that i previously sent to other researchers. These two different origins are represented by the two main

[9] By using the per-period expected payoff to analyze the incentives to deviate from a particular network, we do not consider transition effects from one network to another. This simplification has minor implications, especially for cases in which transition does not last in time and/or the discounting rate is near to one.

parts of (2), respectively. For $g_{ji}=1$, $\Theta(i)\frac{1}{\mu_j}$ can be interpreted as the Steady State probability that an open idea coming from node j is chosen to be processed by i. Given that all ideas in a given stock have the same probability of being selected, this probability is obtained by multiplying the share of ideas coming from j with respect to all ideas received by researcher i ($\frac{\frac{1}{\mu_j}}{\sum_{k\in N_i(g)}\frac{1}{\mu_k}}$) by the expected number of ideas that node i processes per period (1 if $\sum_{k\in N_i(g)}\frac{1}{\mu_k}\geq\frac{q+1}{\rho}$ and o_i otherwise[10]). On the other hand, $cf(h_j+h_i)$ (or $f(h_i)$) can be interpreted as the expected payoff obtained by each of the coauthors of a processed idea.

Notice that there are three main factors affecting agent i's expected utility: c influences the decision between retaining open ideas or sending them to other authors, the queue size (included in $\Theta(l)$ or $\Theta(i)$) affects the probability of processing an open idea, and the coauthor's amount of talent (h_l) affects the probability of publishing processed ideas.

The network stability concept used in this model is *Pairwise-Nash Equilibrium* (PNE hereafter) . In a PNE network, no player must have incentives to deviate unilaterally (*i.e.* the usual Nash Equilibrium condition) but we further require that any mutually beneficial link be formed in equilibrium. PNE networks are robust to bilateral commonly agreed one-link creation and to unilateral deviations. Formally, a pure strategy $s^*=(s_1^*,...,s_n^*)$ is a Nash Equilibrium of the game of network formation (previously described) if and only if $\Pi_i(g(s^*))\geq\Pi_i(g(s_i,s_{-i}^*))$, for all $s_i\in S_i$ and $i\in N$. Let $g+ij$ be the network obtained by adding the link g_{ij} to g.

Definition 1 *A network g is a PNE network with respect to the network payoff function Π if and only if there exists a Nash equilibrium strategy profile s^* that supports g, that is, $g=g(s^*)$, and, for all pair of players i and j such that $g_{ij}=0$ if $\Delta\Pi_i(g+ij)>0$ then $\Delta\Pi_j(g+ij)<0$.*

Given that s_{ii}^i is part of s_i $\forall i\in N$, a deviation consisting on a multi-link severance by i and a simultaneous creation of g_{ii} is an unilateral deviation because there is no need of mutual consent to form g_{ii}.

Notice that this is a relatively weak equilibrium concept[11]. Yet, we are able to isolate a single equilibrium network topology for a specific parameter space.

3 Results

The empirical study by Goyal, Van der Leij and Moraga (2006) is describes a detailed image of the features of real coauthorship networks in Economics. The results of that paper "show that the world of Economics is spanned by inter-linked stars, that this feature is stable over time and that this is the main reason for small average distances". A similar conclusion applied to Biology, Physics and Mathematics can be extracted from Newman (2004). Our aim is to show how the model introduced in the previous section reproduces equilibrium structures with the features of actual

[10]Notice that for a given g, q and ρ the expected stock of open ideas is determined by (1).

[11]The concept of Bilateral Equilibrium (also called pairwise stable Equilibrium) introduced in Goyal and Joshi (2003) and used in Van der Leij (2006) is much stricter than PNE since it allows pairs of players to form and delete links simultaneously.

scientific collaboration networks. In reality, these networks have a highly unequal distribution of links with a small fraction of researchers concentrating many links. In our model, this feature implies that some nodes receive many open ideas from others and, consequently, they are able to publish more papers than other researchers. Therefore, the payoffs of those nodes that receive a large proportion of open ideas are significantly larger than the payoffs of nodes that receive a small proportion of ideas. Reproducing such an unequal payoff distribution in the equilibrium of a setting in which players can discretionally create or sever links is relevant when the population is fairly homogeneous, *i.e.* when talent is fairly constant across agents. In spite of that, our results show that for sufficiently low values of ρ (with respect to the processing rate which is one), these networks naturally arise from the interaction among similar agents. Evidently, the more heterogeneous population is, the easier it is to reproduce these unequal distributions of links in equilibrium and, as a consequence, the requirements on ρ are less stringent.

The section is divided into two parts. First, we present some preliminary results that determine the basic characteristics of equilibrium networks. We derive a Corollary that highlights how networks with a highly unequal distribution of links can be sustained in a PNE. This result shows that, in spite of having very low *a priori* differences among agents' talents, this kind of topologies can be PNE networks if ρ is sufficiently low. In these equilibrium networks high talent researchers receive an amount of links sufficient to exhaust their processing capability but sufficiently small to not suffer congestion. The second subsection shows that a particular interlinked star network, that holds the previously stated conditions, can naturally arise –from the interaction of self-interested researchers– as the unique PNE structure for any distribution of talent. Heterogeneity and congestion are the key factors that explain these results.

Before presenting the results, we comment the requirements about ρ. This paper shows that equilibrium networks have the form of an interlinked star when ρ is sufficiently small with respect to their processing capability, *i.e.* when agents receive a low number of ideas per period from outside the network in relation to their processing capability. Notice that the higher it is the heterogeneity among agents the less stringent it is the restriction on ρ for results to hold. We emphasize that this is not an unrealistic requirement. In real coauthorship networks, the most connected researchers have around 50 links in Economics (as shown by GVM) and even more in other fields as Biology (Newman, 2004). In our model, this means that each particular collaboration link contributes a small quantity of open ideas to their destination with respect to the processing rate. For this reason, researchers have to maintain the link with many collaborators in order to receive a sufficient amount of ideas to exhaust most of their processing capability.

3.1 Preliminary results

In the following three propositions we show that, when ρ is sufficiently low, PNE networks must hold several characteristics for any possible distribution of talents (h). In particular, these results establish an upper-bound in the number of out-degree and in-degree links of any node and a lower bound in the number of in-degree links of nodes with a relatively high level of talent. Therefore, they narrow the set of potential PNE networks. In each of the results we highlight that the higher it is the heterogeneity among agents the less stringent can be the requirements on ρ.

The first result refers to the number of out-degree links of a particular agent. In principle, the researchers of this model can send their open ideas to many collaborators, *i.e.* there is no upper-bound on μ_i $\forall i$. The following result approaches this issue.

Let $G^*_{h,\rho}$ be the set of PNE networks for a given pair (h, ρ).

Proposition 1 *For any functional form of $f(\cdot)$, for any pair (h, ρ_0) and any $g \in G^*_{h,\rho_0}$ in which $\mu_i > 1$ for some $i \in N$, there always exists a $\bar{\rho}_1 < \rho_0$ such that $g \notin G^*_{h,\rho}$ $\forall \rho < \bar{\rho}_1$.*

The proof (see in Appendix) proceeds as follows. We consider all possible cases that a researcher i with $\mu_i \geq 2$ can face. Then, we analyze her incentives to sever the link g_{ij} where $h_j < h_k$ for some $k \in M_i$. We show that, for any vector h whose values are extracted from a continuous distribution function, the marginal payoff derived from such a deviation tends to be positive as $\rho \to 0$. Using the concept of limit, we show how this is a reformulation of the statement of the proposition.

The intuition of the proof is simple. Heterogeneity implies that any agent i can rank the destinations of her open ideas with respect to their level of talent. In other words, i can rank all the agents in M_i with respect to the expected payoff obtained from the processed ideas transmitted to them. If i does not send all open ideas to the destination that maximizes this expected payoff, is because of another important factor: agents have limited processing capability. When an agent severs a link, she automatically increases the flow of open ideas sent to the rest of destinations and, consequently, increases the destinations' queue of ideas when congestion arises. This has a negative impact on the expected utility of the agent who initially deviates. In the proof we show that as $\rho \to 0$ the negative impact tends to vanish. Consequently, for a sufficiently low value of ρ, any agent i with $\mu_i \geq 2$ has incentives to deviate and send all open ideas to the most talented destination (who has a talent $\bar{h}$). The higher it is $\bar{h}$ with respect to the talent of the rest of destinations, the higher would be i's incentives to deviate and, therefore, the higher $\bar{\rho}_1$ can be.

This result illustrates that whatever h, we can always find a sufficiently low value of ρ (say $\bar{\rho}_1$) such that a network in which some agent has two or more out-degree links cannot be sustained in equilibrium for any $\rho < \bar{\rho}_1$. Moreover, increasing the heterogeneity among agents' talent can make the restriction on ρ less stringent. Therefore, an upper-bound on the equilibrium out-degree of nodes arises naturally. As introduced above, this result implies a dramatic simplification of the set of possible PNE networks.

Focusing on the in-degree of nodes we can go one step further in the simplification. The following result establishes an upper-bound for the amount of ideas (and indirectly the amount of links) a node can receive in a PNE network when ρ is sufficiently small.

Proposition 2 *For any functional form of $f(\cdot)$, for any pair (h, ρ_0) and any $g \in G^*_{h,\rho_0}$ in which $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$ for some $i \in N$, there always exists a $\bar{\rho}_2 < \rho_0$ such that $g \notin G^*_{h,\rho}$ $\forall \rho < \bar{\rho}_2$.*

The intuition of the proof (see in Appendix) is the following: when $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$ for some $i \in N$, the stock of open ideas of i will be higher than one, even after severing one in-degree link.

Given that the maximum processing rate is normalized to one, researcher i can delete one in-degree link without damaging the average processing flow (which will continue to be one). Consequently, agent i can increase her average productivity if she severs an in-degree link coming from a researcher who holds two conditions: (i) her talent is below the average talent of the rest of agents in N_i and (ii) she does not receive a link from i. From Proposition 1 we know that for any $\rho < \bar{\rho}_1$, agent i will send all her open ideas to a unique destination. Then, in order to hold the two conditions above, we only need to have two agents below the average talent of the rest of agents in N_i. Evidently, this is easier when there is a high level of heterogeneity among the agents in N_i and i has a large number of in-degree links. In the proof of this proposition we show that for any arbitrarily homogeneous distribution of talents there always exists a $\bar{\rho}_2$ that assures the existence of such a pair of collaborators for any $\rho < \bar{\rho}_2$ and therefore assures that no researcher i can hold $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$.

The form of the production function has a direct effect on the incentives of collaboration. For a concave $f(\cdot)$, working with another researcher (rather than alone) increases the probability of publication less than proportionally with respect to the increase in the amount of talent. On the other hand, a convex $f(\cdot)$ implies that adding additional talent in the production process increases the publication probability more than proportionally. For this reason, we can say that a concave $f(\cdot)$ discourages agents to look for collaborators. Previous results are valid for any functional form of $f(\cdot)$. Now, we focus on the case in which $f(\cdot)$ is linear or convex. In that way we specifically analyze the features of equilibrium networks in the case in which collaborating with other researchers does not imply a loss of productivity with respect to working alone. The results for this case narrow the set of potential PNE networks even more.

Proposition 3 *For a linear or convex $f(\cdot)$, a PNE network cannot have a player i with $\sum_{l \in N_i} \frac{1}{\mu_l} < \frac{q+1}{\rho} - 1$ when some agent j such that $h_j < h_i$ holds: $g_{jk} = 0$ $\forall k$ such that $h_k \geq h_i$.*

See proof in Appendix. When $f(\cdot)$ is linear or convex and without considering the effects of a potential congestion, researchers always prefer to have collaborators with higher talent. If $\sum_{l \in N_i} \frac{1}{\mu_l} < \frac{q+1}{\rho} - 1$ for some $i \in N$, agent i will not suffer congestion even after receiving a new link. For this reason, if all the researchers that receive ideas from another researcher j have a talent lower than h_i, then agent j has incentives to create a new link to agent i. Moreover, since i will not congest after receiving this new link, she will increase her processing rate and, therefore, her payoff will also increase. Thus, such pair of agents cannot exist in a PNE network. Again, heterogeneity and congestion are essential for shaping this result.

Considering this result, and Propositions 1 and 2 we can state the following:

Corollary 1 *For a linear or convex $f(\cdot)$ and for any vector h, highly talented researchers receive all links of the network in any $g \in G^*_{h,\rho}$ for any $\rho < \bar{\rho}_2$. Specifically, $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_i(g)} \frac{1}{\mu_l} < 1 + \frac{q+1}{\rho}$ for a highly talented researcher i and any $\rho < \bar{\rho}_2$.*

The first claim of the corollary directly derives from Propositions 1 and 3. The bounds of $\sum_{l \in N_i(g)} \frac{1}{\mu_l}$ are obtained in Propositions 2 and 3. This result illustrates the attraction force of highly talented researchers in this model. They will always have a minimum number of collaborators. On the other hand, this attraction force is restricted by the possibility of congestion derived from the agents' limited processing capability. For this reason, there is an upper-bound in the number of collaborators a node can have in equilibrium. We must also remember that the higher it is the heterogeneity among the levels of talent the higher $\bar{\rho}_2$ can be.

Notice that if $\sum_{l \in N_i} \frac{1}{\mu_l} = \frac{q+1}{\rho}$ for some $i \in N$, the stock of open ideas of agent i is one, which is exactly her processing rate. Then, this can be seen as the point in which agents receive the minimum number of ideas in order to process at their maximum rate. At that point agents maximize their processing rate without suffering congestion. The bounds established in Corollary 1 imply that in a PNE network highly talented researchers will have a number of in-degree links that allow them to be close to exhausting their processing capability without suffering congestion.

Notice also that (for a linear or convex $f(\cdot)$) there is no heterogeneity requirement on the distribution of talents to obtain Proposition 3. So, the attraction force of the best researchers exists regardless of the difference between their level of talent and that of the other researchers. That is, the result holds for any vector h extracted from a continuous distribution function.

Since $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_i(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$, note that the lower it is ρ the higher it is $\sum_{l \in N_i(g)} \frac{1}{\mu_l}$. Consequently:

Corollary 2 *For a linear or convex $f(\cdot)$, arbitrarily small differences among agents' talent can generate highly unequal distributions of links in the PNE network candidates. The lower it is ρ, the higher it is the in-degree inequality.*

Moreover, because of the convexity of the relationship between ρ and $\sum_{l \in N_i(g)} \frac{1}{\mu_l}$, very small changes in ρ translate into large increases in the in-degree inequality among researchers. This contrasts with the results of Van der Leij (06) in which a minimum degree of heterogeneity among agents is required in order to reproduce the empirical results about in-degree inequality.

So, we have reproduced an empirical fact. In our Steady State network, some few agents can concentrate many links (especially for low values of ρ). But, does this mean that the network we obtain in equilibrium is an interlinked star as suggested by the empirical results of Goyal, Van der Leij and Moraga (2006)? By the previous results, it is not necessarily the case. For example, we can have a network formed by stars, in which the central agents (who receive a number of links respecting the bounds established in Corollary 1) are not connected between them. The following results focus on showing that for any arbitrarily homogeneous distribution of talents there exist a sufficiently low ρ under which this cannot happen. Yet, in the following subsection we go one step further and characterize a particular interlinked star network as the unique PNE for low values of ρ. Again, the more heterogeneous it is the distribution of talents the less stringent it is the condition on ρ.

3.2 Main results

First, we will characterize our PNE network candidate by developing some preliminary steps. Let G^s denote the set of interlinked star networks holding the following three properties:

- $\mu_i = 1 \; \forall i \in N$.
- $g_{ii} = 1$ if and only if $h_i > h_l \; \forall l \in N$ and $l \neq i$.
- For any given pair (h_i, h_j) such that $h_i > h_j$ and $g_{ik} = g_{jl} = 1$ for any $k, l \in N$, it must be true that $h_k \geq h_l$.

The first condition states that all the nodes of the network have only one out-degree link. The second property implies that, except for the highest talented player, this links to some other agent. Finally, the third condition narrows the set of possible destinations of this out-degree link. In fact, the last two conditions imply that the highest talented agent receives her in-degree links from the nodes located just below her in the ranking of talents; the second player in this ranking receives her in-degree links from the nodes located just below the first group of players in this ranking; and so on. Depending on the distribution of links we can have different networks in G^s. The following result specifies that, when $\rho < \bar{\rho}_2$, there is a single distribution of links (*i.e.* a single network) in G^s holding two stability conditions.

Lemma 1 *For any given pair (h, ρ) such that $\rho < \bar{\rho}_2$, there is a unique network in G^s in which no agent i with $\eta_i > 0$ has incentives either to delete any in-degree link or to create a new in-degree link coming from some j such that $h_j < h_i$. We call this network g^s.*

See the proof in Appendix. Notice that since $g^s \in G^s$, (i) this graph is an interlinked star network, that is, there is a group of authors linked among them that concentrate certain number of links and (ii) the coauthors of a given researcher do not collaborate with each other. This is, in essence, the basic topology of actual scientific collaboration networks according to the empirical studies commented above. The following two results show that g^s is a natural equilibrium outcome of our model.

Proposition 4 *For a linear or convex $f(\cdot)$, for any pair (h, ρ_0) and any $g \in G^*_{h,\rho_0}$ different from g^s, there always exists a $\hat{\rho}_1 < \rho_0$ such that $g \notin G^*_{h,\rho} \; \forall \rho < \hat{\rho}_1$.*

See the proof in the Appendix. This result shows that whatever it is h, we can always find a sufficiently low value for ρ under which a network different from g^s cannot be sustained as a PNE. Therefore, the set of potential PNE networks for a sufficiently small ρ reduces to one single topology. In the proof we show that for any network $g \neq g^s$ we can always find a player (or a pair of players) whose marginal payoff for deviating tends to be positive as $\rho \to 0$. Notice that, using the definition of limit, this implies that such a network g cannot be PNE for any arbitrarily small ρ. Intuitively, the mechanisms underlying this result are the following. We first show that in a network different from g^s and with $\rho < \bar{\rho}_1$, there must exist an agent i who sends all her ideas to a node (say k) such that $h_k < h_j$, where j is the node that should receive ideas from i in g^s. If i does not deviate from this network by creating a new link g_{ij} is because of the possibility that j has a longer queue than k due to congestion. As ρ approaches zero the differences between agents' queue sizes (for example, j and k) become relatively smaller. We reach a point in which player i's incentives to deviate are basically driven by the differences between the talents of j and k. Since $h_k < h_j$, player i would have incentives to add g_{ij}. Evidently, the higher it is h_j relative to h_k the more incentives player i has to deviate. Therefore, the less stringent the condition on ρ would be. Then, a higher heterogeneity of talents can increase agent i's incentives to deviate.

However, for the deviation to take place, node j should have incentives to form the new link g_{ij}. In the worse case, agent j would only accept such a link if the average productivity from the ideas of her queue increases after the deviation. For a sufficiently low ρ, we can state that as $\rho \to 0$, this average productivity decreases. Then, there is a point in which this average is sufficiently low that the formation of g_{ij} pushes this average productivity up. Again, it is easy to show that a higher heterogeneity among the levels of talent would increase agent j's incentives to deviate.

After Proposition 4, the set of PNE candidates reduces to one single network for a sufficiently low ρ. The following result confirms that g^s is in fact a PNE for any arbitrarily homogeneous distribution of talents whenever ρ is sufficiently small.

Proposition 5 *For a linear or convex $f(\cdot)$ and for any pair (h, ρ_0), if $g^s \notin G^*_{h,\rho_0}$ there always exists a $\hat{\rho}_2 < \rho_0$ such that $g^s \in G^*_{h,\rho}$, $\forall \rho < \hat{\rho}_2$.*

See the proof in the Appendix. In the proof we review all the possible deviations from g^s. We find that, in the worse cases, the marginal payoff of potential deviators tends to be negative as $\rho \to 0$. Therefore for any vector h, we can always find a sufficiently low ρ such that no player has incentives to deviate from g^s. From an intuitive point of view, the proof can be explained as follows. There are two kinds of deviations from g^s. On the one hand, agent i can create an additional link to an agent with a lower talent than that of the previous destination or she can substitute her current out-degree link by g_{ii}. In both cases, this agent trades-off the potential benefits from avoiding or reducing the effects of congestion against the costs of reducing the productivity of the processed ideas due to the lower talent of the new destination. In the proof we show that the positive part of this trade-off tends to vanish as $\rho \to 0$, and in consequence, the marginal payoff for deviating tends to be negative. Again, increasing heterogeneity among agents would increase the differences in talent and thus, increase the negative part of the commented trade-off. As a consequence, $\hat{\rho}_2$ can be higher as we increase the differences among agents' talent. The other type of deviation consists of creating an additional link to an agent with a talent higher than that of the previous destination. In that case, we use Lemma 1 to conclude that the marginal payoff of the new destination to accept the link is negative.

The following corollary immediately emerges from the last two propositions. Let $\hat{\rho} \equiv min(\hat{\rho}_1, \hat{\rho}_2)$.

Corollary 3 *For a linear or convex $f(\cdot)$ and for any vector h, there exists a $\hat{\rho}$ such that g^s is the unique PNE network for any $\rho < \hat{\rho}$.*

Existence of g^s as a PNE comes from Proposition 5 and uniqueness comes from Proposition 4.

Notice again, that we do not need to impose any degree of heterogeneity among agents. Specifically any talent vector h extracted from a continuous distribution function can generate the previous result. Nevertheless, the higher it is the heterogeneity among agents the less stringent it can be the condition on the parameter ρ in order to get g^s as the unique PNE network. For these reasons we conclude that the kind of networks GVM observes in reality are a natural outcome from the interaction of self-interested researchers of this model.

Before ending this section of results, it is interesting to provide hints about the behavior of the model for a concave $f(\cdot)$. As commented above, concavity of $f(\cdot)$ discourages agents at the moment of looking for collaborations. Consequently, we cannot assure the stability of the interlinked star for all distributions of talent even for an arbitrarily small ρ. In particular, we would need a sufficiently high inequality across levels of talent to support such a network of collaborations. This inequality is also needed to support other structures with high in-degree inequality (non-interlinked stars) as stable networks.

Remark 1 *For $f(\cdot)$ concave, networks with high in-degree inequality (as an interlinked star) may not be PNE even for arbitrarily small ρ. To assure stability of this kind of networks we need a minimum degree of heterogeneity among researchers' talents.*

Other networks such as the empty network or the cycles can exist in equilibrium even with low values of ρ, especially when inequality across levels of talent is not so high. Summarizing, for a concave $f(\cdot)$ the key factor affecting the shape of the stable network is the inequality across levels of talent. Only highly unequal talent distributions will allow obtaining stable networks with a high in-degree inequality such as the interlinked star.

4 Discussion

4.1 Empirical patterns

Based on empirical patterns, Goyal, Van der Leij and Moraga (2004) reach the conclusion that the field of economic research is spanned by interlinked stars. In this paper we showed that a simple network formation model characterized by the limited processing capability of heterogeneous agents can reproduce the characteristics of the in-degree distribution of the so called interlinked star network in equilibrium. In this section, we discuss how our model may be extended to explain other empirical patterns.

One of the first empirical findings by GVM relates to the average number of collaborators. The giant component of the analyzed coauthorship network reveals an average of 2.48 in the 1970's and 3.06 in the 1990's[12]. None of the results of our model excludes the possibility of having these average numbers of collaborators. In fact, we can have equilibrium networks with 2, 3 or more collaborators per researcher. But, as Proposition 1 shows for low values of ρ, it would be especially difficult to have a researcher $i \in N$ with more than one out-degree link in a PNE network. Therefore, for low values of ρ our model can hardly reproduce this average number of collaborators.

[12] Newman (2004) founds that the average number of collaborators in Biology, Physics and Mathematics was 18.1, 9.7 and 3.9 respectively.

A natural extension of the model will allow us to reproduce this empirical fact. Imagine a model in which there exist different types of talent and researchers are specialists, so they have a specific type. Moreover, ideas can require some specific type of talent to be published that does not necessarily coincide with the type of talent of the first receiver. For this reason, we can also classify the ideas on different types. To be specific let h_i^x denote the agent i's amount of talent of type x. Let $f_x(h_i^x + h_j^x)$ be the expected probability of publishing a processed idea of type x by researchers i and j. The rest of the model would not change with respect to the model described in section 2. In this new model, agents would have incentives to select specialist collaborators for each of the different types of ideas they can receive. Thus, equilibrium networks would be able to reproduce higher average numbers of collaborators. Moreover the average number of collaborators in equilibrium would positively depend on the degree of researchers' specialization. Therefore the model would be a formalization of the argument defending that one of the key factors explaining the increase in the flow of scientific collaboration is the increase in the specialization of researchers. This increase in the number of collaborators is a trend that GVM detected for the last 30 years in the field of economic research.

With respect to the degree distribution, GVM finds that such a distribution exhibits fat-tails, with a small fraction of scientists having a large number of collaborators. The same can be concluded for the fields of Biology, Physics and Mathematics, as Newman (2004) shows. Our model shows that the links concentrate in highly talented researchers. Decreasing the value of ρ increases the number of links directed towards each of these researchers and, in consequence, increases the inequality in the in-degree distribution. In equilibrium our model predicts that highly talented researchers have roughly the same number of links (see Corollary 1). Evidently, this is not the case in actual networks. This result arises because we assume that all players in our game have exactly the same processing capability, which we have normalized to 1. By allowing different processing rates, the model is able to reproduce equilibria with different in-degree levels for different players.

The last empirical pattern we discuss refers to clustering. GVM shows that *"the most connected individuals collaborated extensively and most of their coauthors did not collaborate with each other"*. Our model cannot provide an intuitive explanation to GVM's results. By focusing on the role of heterogeneity among players and their limited processing capability (as our model does), we can only provide an intuitive argument for explaining the number of links per node which is not a sufficient condition for explaining their level of clustering. In order to explain clustering, we would have to consider geographic or conceptual proximity among researchers. A simple extension of the model can capture this consideration: let us assume that researchers are distributed in broadly-defined groups. Two researchers can be members of the same group if they are in the same academic department, if they work on similar topics or if they share a common personal characteristic. If two researchers i and j are in the same group then $d_{ij} = 1$; otherwise $d_{ij} = 0$. We can reasonably argue that the collaboration between two members of the same group will be more productive than the collaboration between two more distant researchers. Formally, we can write the expected probability of publishing a processed idea by nodes i and j as $f(h_i + h_j + kd_{ij})$ for some $k > 0$. This simple extension allows the model to reproduce equilibrium networks with high clustering among members of the same group and low clustering among highly talented researchers of different groups. Rubí-Barceló (2007) analyzes a network formation model with *a priori* distributions of agents[13].

4.2 Stability and efficiency

A network is efficient when it maximizes the aggregate payoff. The set of efficient networks usually does not coincide with the set of stable networks. In fact, one of the most usual analyses in the network formation models is the comparison between stable and efficient networks. Jackson (2004) collects a variety of examples. Jackson and Wolinsky (1996) develops a simple coauthorship network formation model as an example to show how negative externalities can play an important role in the network formation process in the academic world, and in particular, in the conflict between stability and efficiency. In the efficient network that the model yields, researchers are distributed in pairs, *i.e.* two agents connected with each other and isolated from the rest. However, the stable network is over connected with respect to the efficient network. In that model, the advantageous strategy from an individualistic point of view does not coincide with the good strategy from an aggregate point of view. This phenomenon, is especially relevant in coauthorship networks because it implies that researchers' individual incentives damage aggregate scientific production. In this sense, Jackson and Wolinsky (1996) offers a pessimistic view of actual scientific collaboration networks. In the following lines we show that in our model, individual and aggregate incentives are more aligned than in Jackson's and Wolinsky's (1996) model. Yet, they do not fully coincide.

The results of Section 3 suggests that g^s is a stable network for a sufficiently small ρ when $f(\cdot)$ is linear or convex. In this section we derive the efficient network(s) for this specific case. This allows us to compare stable and efficient networks. *A priori*, we can say that g^s has favorable characteristics to maximize the aggregate payoff. In particular, for a linear or convex $f(\cdot)$, it seems appropriate that highly talented researchers collaborate with each other. But, is g^s the best structure of collaborations in order to maximize the aggregate payoff for any given pair (h, ρ)? The following result gives an important insight to answer this question.

Proposition 6 *For $g = g^s$ and for any vector h, there exists a ρ^* such that for any $\rho < \rho^*$, if we substitute a link g_{ij} by a new link g_{ik} then the marginal aggregate payoff decreases when $h_k < h_j$ and increases when $h_k > h_j$.*

See the proof in the Appendix. Independently of h, the accumulation of links to highly talented agents has positive implications for the aggregate payoff when ρ is sufficiently small. Evidently, the higher it is the difference in talent between highly talented researchers and the rest of researchers, the higher the aggregate incentives to accumulate links to them, and the less stringent is the requirement on ρ to hold the last proposition.

Once again, the trade off between the benefits of working with highly talented researchers and the costs of working with more congested coauthors come on stage. By changing g_{ij} by a new link g_{ik} such that $h_k < h_j$, the ideas of agent i can avoid congestion problems but they will have a lower

[13] Rubí-Barceló (2007) adopt a different payoff function than the one used here.

probability of publication once processed. On the other hand, by changing g_{ij} by a new link g_{ik} such that $h_k > h_j$, k can suffer congestion problems but i increases the probability of publication of her processed ideas. But when $\rho \to 0$, congestion tends to disappear. Consequently, the aggregate marginal payoff increases if highly talented players receive more ideas and decreases when highly talented players receive less ideas.

The result shows that the PNE interlinked star network we obtained in the previous section is not efficient for any pair (h, ρ). It also shows that the way of increasing efficiency when the entrance rate of open ideas is sufficiently low and the differences among the levels of talent are sufficiently large, is accumulating more links to highly talented researchers than what would be individually desirable.

5 Conclusion

In spite of the large body of empirical research about scientific collaboration networks, there is a lack of foundational theoretical models that analyze how individual decisions contribute to scientific collaboration network formation. This paper proposes a dynamic model to analyze the formation of this kind of networks.

We focus on heterogeneity among agents' productivity and congestion derived from agent's limited processing capability. We show that self-interested researchers in this setting organize themselves in inter-linked stars, as it is suggested by empirical evidence.

References

Albert, R. and A.-L. Barabási (2002). Statistical mechanics of complex networks. *American Physical Society 74*(1), 47–97.

Arenas, A., A. Cabrales, L. Danon, A. Díaz-Guilera, R. Guimerà, and F. Vega-Redondo (2003). Optimal information transmission in organizations: Search and congestion. Working paper UPF 698.

Calvó-Armengol, A. and M. O. Jackson (2004, June). The effects of social networks on employment and inequality. *American Economic Review 94*(3), 426–454.

Cooper, C. and A. Frieze (2003). A general model of web graphs. *Random Structures and Algorithms 22*(3), 311 – 335.

Goyal, S., M. V. der Leij, and J. L. Moraga-González (2006). Economics: An emerging small world. *Journal of Political Economy 114*, 403–412.

Goyal, S. and S. Yoshi (2006). Unequal connections. *International Journal of Game Theory 34*, 319–349.

Hudson, J. (1996). Trends in multi-authored papers in economics. *Journal of Economic Perspectives 10*(3), 153–58.

Jackson, M. (2004). A survey of models of network formation: Stability and efficiency. In G. Demange and M. Wooders (Eds.), *Group Formation in Economics; Networks, Clubs and Coalitions*, Chapter 1. Cambridge University Press.

Jackson, M. O. and B. W. Rogers (2006). Meeting strangers and friends of friends: How random are social networks? *forthcoming American Economic Review*.

Jackson, M. O. and A. Wolinsky (1996). A strategic model of economic and social networks. *Journal of Economic Theory 71*(1), 44–74.

Kranton, R. and D. Minehart (2001, June). A theory of buyer-seller networks. *American Economic Review 91*(3), 485–508.

Laband, D. N. and R. D. Tollison (2000). Intelectual collaboration. *Journal of Political Economy 108*(3), 632–662.

Milgram, S. (1967). The small world problem. *Psychology Today 2*, 60–67.

Newman, M. E. J. (2001a). Scientific collaboration networks I: Network construction and fundamental results. *American Physical Society 64*(1).

Newman, M. E. J. (2001b). Scientific collaboration networks II: Shortest paths, weighted networks, and centrality. *American Physical Society 64*(1).

Newman, M. E. J. (2003). The structure and function of complex networks. *SIAM Review 45*(2), 167–256.

Newman, M. E. J. (2004). Coauthorship networks and patterns of scientific collaboration. *PNAS 101*(suppl. 1), 5200–5205.

Price, D. J. d. S. (1976). A general theory of bibliometric and other cumulative advantage processes. *Journal of the American Society for Information Science 27*, 292–306.

Rubí-Barceló, A. (2006). Can you really get rich because of a structural hole? mimeo. Universitat Pompeu Fabra.

Van der Leij, M. (2006). *The Economics of Networks: Theory and Empirics*. Ph. D. thesis, Tinbergen Institute. Erasmus University.

Watts, D. (1999). *Small Worlds*. Princeton University Press.

A Proofs

Proof of Proposition 1. Let g be a PNE for a given pair (h, ρ_0). Let $i \in N$ be a member of this network who has $\mu_i \geq 2$. We claim that there exists a $\bar{\rho}_1 < \rho_0$ such that g will not be PNE for any pair (h, ρ) such that $\rho < \bar{\rho}_1$.

Let us focus on the case in which $\mu_i = 2$ (the cases in which $\mu_i > 2$ can be proved analogously). Let players k and j be the destinations of these two links, that is, $g_{ij} = g_{ik} = 1$. Here we consider the case in which $i \neq j$ and $i \neq k$. The case in which $i = j$ or $i = k$ is analogous, thus omitted. Next we analyze i's incentives to deviate. At this point we have to distinguish between the following two cases:

i $\sum_{l \in N_r(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ for $r = k, j$.

The i's marginal payoff for cutting the link g_{ij} off is positive if and only if:

$$\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}[\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l}} + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}] \tag{i}$$

On the other hand, i's marginal payoff for cutting the link g_{ik} off is positive if and only if:

$$\frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}[\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l}} + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}] \tag{ii}$$

Given that $0 < \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l}} \leq \frac{\rho}{q+1}$ for $r = k, j$ we can say that:

$$lim_{\rho \to 0}(\frac{1}{\sum_{l \in N_r} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l}}) = 0$$

Following the definition of limit we can say that for any $\varepsilon > 0$, there exists a ρ' such that $|\frac{1}{\sum_{l \in N_r} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l}}| < \varepsilon$ for $r = k, j$, for any $\rho < \rho'$. Given that $h_j \neq h_k$, notice that if the LHS of conditions (i) and (ii) were equal to $\frac{f(h_i + h_r)}{\sum_{l \in N_r} \frac{1}{\mu_l}}$ for $r = k, j$ respectively, (i) and (ii) would be complementary. So, we conclude that for any specific vector h, and in particular for any specific triple (h_i, h_j, h_k), there exists a ρ' such that some of the two conditions (i) and (ii) has to hold for any $\rho < \rho'$. That is, for a sufficiently small ρ agent i will have incentives to deviate and cut one of her out-degree links off.

ii $\sum_{l \in N_j(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ and $\frac{q+1}{\rho} - \frac{1}{2} \leq \sum_{l \in N_k(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ (or vice versa).

The i's marginal payoff for cutting the link g_{ij} off is positive if and only if:

$$\frac{f(h_i + h_k)}{\sum_{l \in N_k} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}[\frac{\rho}{1+q} f(h_i + h_k) + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}] \tag{iii}$$

The i's marginal payoff for cutting the link g_{ik} off is positive if and only if:

$$\frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{2}} > \frac{1}{2}[\frac{\rho}{1+q} f(h_i + h_k) + \frac{f(h_i + h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}] \tag{iv}$$

Given that $0 < \frac{1}{\sum_{l \in N_r(g)} \frac{1}{\mu_l} + \frac{1}{2}} \leq \frac{\rho}{q+1}$ for $r = k, j$ we can say that $lim_{\rho \to 0}(\frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{\rho}{1+q}) = 0$ and $lim_{\rho \to 0}(\frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}}) = 0$. Following the definition of limit we can say that for any $\varepsilon > 0$, there exists a ρ'' such that $|\frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{\rho}{1+q}| < \varepsilon$ and $|\frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{2}} - \frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}}| < \varepsilon$ for any $\rho < \rho''$. Given that $h_j \neq h_k$, notice that if the LHS of conditions (iii) and (iv) were equal to $\frac{\rho f(h_i+h_k)}{1+q}$ and $\frac{f(h_i+h_j)}{\sum_{l \in N_j} \frac{1}{\mu_l}}$ respectively, (iii) and (iv) would be complementary. So, we conclude that for any specific vector h, and in particular for any specific triple (h_i, h_j, h_k), there exists a ρ'' such that some of the two conditions (iii) and (iv) has to hold for any $\rho < \rho''$. That is, for a sufficiently small ρ agent i will have incentives to deviate and cut one of her out-degree links off.

iii $\frac{q+1}{\rho} - \frac{1}{2} \leq \sum_{l \in N_r(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ for $r = k, j$.

iv $\sum_{l \in N_j(g)} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ and $\sum_{l \in N_k(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} - \frac{1}{2}$.

v $\frac{q+1}{\rho} - \frac{1}{2} \leq \sum_{l \in N_j(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ and $\sum_{l \in N_k(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} - \frac{1}{2}$.

The proof for these three cases proceeds analogously to the previous one.

vi $\sum_{l \in N_r(g)} \frac{1}{\mu_l} < \frac{q+1}{\rho} - \frac{1}{2}$ for $r = k, j$.

In this case it is easy to observe that agent i will have incentives to sever the link with the agent with the lowest level of talent for any given ρ.

Therefore, in any possible case in which $\exists i \in N$ such that $\mu_i \geq 2$, we can find a sufficiently low value for ρ under which there is a profitable deviation. Defining $\bar{\rho}_1$ as the minimum of all these values of ρ, the proof of the proposition is done. ■

Proof of Proposition 2. Suppose we have an agent i such that $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho_0}$ in a network $g \in G^*_{h,\rho_0}$ for a given pair (h, ρ_0). First, we claim that there is a ρ (say $\bar{\rho}_2$) such that for any $\rho < \bar{\rho}_2$, $\exists k \in N_i$ such that $k \neq i$, $k \notin M_i$ and:

$$cf(h_i + h_k) < \frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} (\sum_{l \in N_i \setminus \{i,k\}} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii} \frac{1}{\mu_i} f(h_i)) \qquad \text{(v)}$$

In words, for a sufficiently low ρ, some player k with a link towards i ($k \neq i$) has a joint productivity with i below the average productivity of i without considering researcher k. Notice that from Proposition 1, there exists a ρ (that we call $\bar{\rho}_1$) such that $\mu_i = 1$ in any $g \in G^*_{h,\rho}$ for any $\rho < \bar{\rho}_1$. Therefore, if at least two agents $j, k \in N_i$ have a joint productivity with i below the average of the rest of players in N_i, we know that one of them will not be in M_i for any $\rho < \bar{\rho}_1$, thus one of the players holds the conditions stated above. However, we can formulate a counter example. Imagine there is a single agent (say j) whose joint productivity with i is below the average of the rest of players in N_i. Moreover, let $j \in M_i$. In such a case, there exists no player k holding the conditions stated above. Next, we show that for a sufficiently low ρ this case cannot exist when $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho}$.

Consider an $\varepsilon > 0$ arbitrarily small. Imagine that $cf(h_j + h_i) = \varepsilon$ for $j \neq i$ and $f(h_j) = \varepsilon$ for $j = i$. Let $k \in N_i$, $k \neq j$ and:

$$\begin{cases} \varepsilon < cf(h_k + h_i) < cf(h_l + h_i)\ \forall l \in N_i \backslash \{i, k, j\} \text{ and } cf(h_k + h_i) < f(h_i) & \text{for } k \neq i \\ \varepsilon < f(h_k) < cf(h_l + h_i)\ \forall l \in N_i \backslash \{k, j\} & \text{for } k = i \end{cases}$$

Let us assume that $cf(h_k + h_i) = b$ for $k \neq i$ (or $f(k) = b$ for $k = i$). We know that there is an $\varepsilon > 0$ arbitrarily small under which $cf(h_l + h_i) > b + \varepsilon\ \forall l \in N_i \backslash \{i, j, k\}$.Then, the average productivity of the players of N_i without including agent k is higher than:

$$\frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} (\varepsilon + (\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_j} - \frac{1}{\mu_k})(b + \varepsilon))$$

After simple algebra we conclude that this expression is higher than b when the following condition holds:

$$b + \varepsilon > (b + \frac{b}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}) \frac{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k} - \frac{1}{\mu_j} + 1}$$

Given that $\mu_j \geq 1$, $\frac{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k} - \frac{1}{\mu_j} + 1} \leq 1$. Since $\sum_{l \in N_i} \frac{1}{\mu_l} \geq 1 + \frac{q+1}{\rho}$, we conclude that $lim_{\rho \to o} \frac{b}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} = 0$. This implies that for any $\varepsilon > 0$, we can always find a value of ρ (say ρ') such that $|\frac{b}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}}| < \varepsilon$ for any $\rho < \rho'$. Therefore, for any $\rho < \rho'$ we have at least two agents (j and k) who hold condition (v). As commented above, one of them must fulfill the conditions stated in the beginning of the proof when $\rho < \bar{\rho}_1$. Let $\bar{\rho}_2 \equiv min(\bar{\rho}_1, \rho')$. That concludes the proof of the initial claim.

Our next step is to show that a network that holds these conditions cannot be sustained as a PNE for $\rho < \bar{\rho}_2$.

Agent i's marginal payoff for cutting the link g_{ki} off is:

$$\Delta\Pi_i = \frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l} - \frac{1}{\mu_k}} [\sum_{l \in N_i \backslash \{i,k\}} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii} \frac{1}{\mu_i} f(h_i)] - \frac{1}{\sum_{l \in N_i} \frac{1}{\mu_l}} [\sum_{l \in N_i \backslash \{i\}} \frac{1}{\mu_l} cf(h_l + h_i) + g_{ii} \frac{1}{\mu_i} f(h_i)]$$

After simple algebra we can say that $\Delta\Pi_i > 0$ if and only if condition (v) holds. This happens by definition of k. Thus, agent i has incentives to deviate. ■

Proof of Proposition 3. Before proving this proposition we formulate an additional lemma.

Lemma 2 *If $c \geq \frac{1}{2}$, $f(0) = 0$ and $f(\cdot)$ is linear or convex, then $cf(h_k + h_l) > f(h_l)$ for $h_k > h_l$.*

Proof. *For a linear or convex $f(\cdot)$ with $f(0) = 0$ and $c = \frac{1}{2}$, the inequality $cf(h_k + h_l) > f(h_l)$ reduces to $h_k > h_l$ after simple algebra. For $f(\cdot)$ convex and $c > \frac{1}{2}$, the difference between $cf(h_k + h_l)$ and $f(h_l)$ will be higher; therefore, the inequality of the statement also holds.* ■

By contradiction let us assume that we have a PNE network in which agent i holds $\sum_{l \in N_i} \frac{1}{\mu_l} < \frac{q+1}{\rho} - 1$ and there exists an agent j with $h_j < h_i$ who holds: $g_{jk} = 0\ \forall k$ such that $h_k \geq h_i$. Imagine the deviation in which player j proposes to player i the formation of a new link. In such case, the marginal utility for player j is:

$$\Delta\Pi_j = \frac{1}{\mu_j + 1}[\sum_{l\in M_j\setminus\{j\}} \hat{\Theta}(l)cf(h_l + h_j) + g_{jj}\hat{\Theta}(j)f(h_j) + \frac{\rho}{1+q}cf(h_j + h_i)] - \frac{1}{\mu_j}[\sum_{l\in M_j\setminus\{j\}} \Theta(l)cf(h_l + h_j) + g_{jj}\Theta(j)f(h_j)]$$

where $\hat{\Theta}(l)$ corresponds to the variable $\Theta(l)$ after the deviation. After simple algebra we can write that $\Delta\Pi_j > 0$ if and only if:

$$\frac{\rho}{1+q}cf(h_j + h_i) + \sum_{l\in M_j\setminus\{j\}} (\hat{\Theta}(l) - \Theta(l))cf(h_l + h_j) + g_{jj}(\hat{\Theta}(j) - \Theta(j))f(h_j) > \frac{1}{\mu_j}[\sum_{l\in M_j\setminus\{j\}} \Theta(l)cf(h_l + h_j) + g_{jj}\Theta(j)f(h_j)]$$

On the one hand, by definition we know that $\Theta(l)$ is lower or equal than $\frac{\rho}{1+q}$ for any $l \in N$, and by assumption we also know that $h_l < h_i\ \forall l \in M_j$. On the other hand, by Lemma 2 we can say that for a linear or convex $f(\cdot)$ and $c \geq \frac{1}{2}$, $cf(h_j + h_i) > f(h_j)$. These conditions imply that $\frac{\rho}{1+q}cf(h_j + h_i) > \frac{1}{\mu_j}[\sum_{l\in M_j\setminus\{j\}} \Theta(l)cf(h_l + h_j) + g_{jj}\Theta(j)f(h_j)]$. Given that for any $l \in M_j$, $\hat{\Theta}(l) > \Theta(l)$, we conclude that $\Delta\Pi_j > 0$.

To complete the proof, we need to show that agent i will have incentives to form the link. Her marginal utility from accepting it is:

$$\Delta\Pi_i = \frac{\rho}{1+q}[\sum_{l\in N_i\setminus\{i\}} \frac{1}{\mu_l}cf(h_i + h_l) + g_{ii}\frac{1}{\mu_i}f(h_i) + \frac{1}{\mu_j + 1}cf(h_i + h_j)] + \frac{c}{\mu_i}\sum_{\substack{l\in M_i\setminus\{i\}\\ l\in M_j}} \hat{\Theta}(l)cf(h_l + h_i) - [\frac{\rho}{1+q}[\sum_{l\in N_i\setminus\{i\}} \frac{1}{\mu_l}cf(h_i + h_l) + g_{ii}\frac{1}{\mu_i}f(h_i)] + \frac{c}{\mu_i}\sum_{\substack{l\in M_i\setminus\{i\}\\ l\in M_j}} \Theta(l)cf(h_l + h_i)]$$

where $\hat{\Theta}(l)$ is defined as above. After simple algebra:

$$\Delta\Pi_i = \frac{\rho}{1+q}cf(h_i + h_j) + \frac{c}{\mu_i}\sum_{\substack{l\in M_i\setminus\{i\}\\ l\in M_j}} (\hat{\Theta}(l) - \Theta(l))cf(h_l + h_i)$$

Since $\hat{\Theta}(l) > \Theta(l)$ for any $l \in M_j$, we conclude that $\Delta\Pi_i > 0$, contradicting the initial statement of stability. ■

Proof of Lemma 1. In this proof we show that for any given pair (h, ρ) such that $\rho < \bar{\rho}_2$, if $g \in G^s$ there is only one possible distribution of links under which no player has incentives either to delete an in-degree or to create a new in-degree link coming from a player with a lower talent. This implies that only one network in G^s can hold this condition. To demonstrate this, we prove that any node i must have a given amount of in-degree links to not have incentives to change η_i,

Imagine a player i and a given in-degree $\eta_i > 0$. Given that $\rho < \bar{\rho}_2 \leq \bar{\rho}_1$, players can have at most a single out-degree link in any PNE network (Proposition 1). Then, $\sum_{l \in N_i} \frac{1}{\mu_l} = \eta_i \; \forall i \in N$. Moreover, since $\rho < \bar{\rho}_2$, we can say that $\frac{q+1}{\rho} - 1 \leq \eta_i < \frac{q+1}{\rho} + 1$ (Corollary 1). Given that η_i can only take natural numbers, there are at most two possible values for η_i in any PNE (say $\bar{\eta}$ and $\bar{\eta} - 1$). We claim that for only one of these two values agent i will not have incentives to change her in-degree for a given pair (h, ρ). First, the following must hold:

$$\frac{q+1}{\rho} - 1 \leq \bar{\eta} - 1 < \frac{q+1}{\rho} \leq \bar{\eta} < \frac{q+1}{\rho} + 1$$

Given these inequalities, we conclude that in any $g \in G^s$ no player i with $\eta_i = \bar{\eta}$ will have incentives to accept an additional in-degree link. Moreover, no player i with $\eta_i = \bar{\eta} - 1$ will have incentives to delete one existing in-degree link. Therefore, there are only two possibilities of deviation. An agent i can have incentives to cut one in-degree link off when $\eta_i = \bar{\eta}$ or she can have incentives to accept some additional in-degree link when $\eta_i = \bar{\eta} - 1$. Analyzing the marginal payoff of both deviations, we observe that if one is positive the other must be negative.

Let $\eta_i = \bar{\eta}$ and $g_{ji} = 1$ in a given network g_1.The agent i's marginal payoff for deleting the in-degree link g_{ji} is:

$$\Delta\Pi_i = \frac{\rho}{1+q}[\sum_{l \in N_i(g_1) \setminus \{i,j\}} cf(h_i + h_l) + g_{ii} f(h_i)] - \frac{1}{\bar{\eta}}[\sum_{l \in N_i(g_1) \setminus \{i\}} cf(h_i + h_l) + g_{ii} f(h_i)]$$

After simple algebra, $\Delta\Pi_i > 0$ if and only if:

$$(\frac{\bar{\eta}\rho}{1+q} - 1)[\sum_{l \in N_i(g_1) \setminus \{i,j\}} cf(h_i + h_l) + g_{ii} f(h_i)] > cf(h_i + h_j)$$

On the other hand, let $\eta_i = \bar{\eta} - 1$ and $g_{ji} = 0$ in a given network g_2 for some agent j such that $h_j < h_i$. The agent i's marginal payoff for creating the link g_{ji} is:

$$\Delta\Pi_i = \frac{1}{\bar{\eta}}[\sum_{l \in N_i(g_2) \setminus \{i\}} cf(h_i + h_l) + cf(h_i + h_j) + g_{ii} f(h_i)] - \frac{\rho}{1+q}[\sum_{l \in N_i(g_2) \setminus \{i\}} cf(h_i + h_l) + g_{ii} f(h_i)]$$

After simple algebra, $\Delta\Pi_i > 0$ if and only if:

$$(\frac{\bar{\eta}\rho}{1+q} - 1)[\sum_{l \in N_i(g_2) \setminus \{i\}} cf(h_i + h_l) + g_{ii} f(h_i)] < cf(h_i + h_j)$$

Notice that $\sum_{l \in N_i(g_2) \setminus \{i\}} cf(h_i + h_l) = \sum_{l \in N_i(g_1) \setminus \{i,j\}} cf(h_i + h_l)$. Therefore, one (and only one) of the two previous inequalities will hold for any vector h and any possible pair of players i and j such that $h_j < h_i$. Then, in G^s, any agent i can receive a single number of in-degree links in order to have no incentives neither to cut some in-degree link off nor to add some new in-degree link from an agent with a lower talent. This implies that only a single network in G^s holds these conditions. ■

Proof of Proposition 4. Let g be a PNE network for some pair (h, ρ_0), i.e. $g \in G^*_{h,\rho_0}$. Let g^s_{ij} denote the link g_{ij} in the network g^s ($g^s_{ij} = 1$ if and only if node i have a link towards j in g^s). Imagine that g is different from g^s. This implies that there exists an agent (say i) such that $g_{il} \neq g^s_{il}$ for some $l \in N$. There are various scenarios in which this holds. In the following lines, we show that there always exists a sufficiently low value of ρ (say $\hat{\rho}_1$) under which none of these scenarios can be sustained in a PNE for any $\rho < \hat{\rho}_1$. Let $j \in N$ be such that $g^s_{ij} = 1$.

(a) The first scenario is that agent i has more than one out-degree link. Following Proposition 1, we know that there exists a ρ (we called $\bar{\rho}_1$) such that such a network cannot be sustained as a PNE for any $\rho < \bar{\rho}_1$.

(b) Second, agent i can have a link to k, i.e. $g_{ik} = 1$, and $h_k < h_j$. Now let us show that this second case cannot hold in a PNE network. Two subcases need to be considered:

- $k = i$. Consider that player i creates a new link g_{ij}. Let us consider the extreme (and less favorable) case in which $o_i \leq 1$ and $\sum_{l \in N_j} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$. The agent i's marginal payoff from this deviation is:

$$\Delta\Pi_i = \frac{c}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i + 1}} \frac{1}{\mu_i + 1} f(h_j + h_i) + \frac{\rho}{1+q} [\sum_{l \in N_i \setminus \{i\}} \frac{1}{\mu_l} cf(h_l + h_i) + \frac{1}{\mu_i + 1} f(h_i)] - \frac{\rho}{1+q} [\sum_{l \in N_i \setminus \{i\}} \frac{1}{\mu_l} cf(h_l + h_i) + \frac{1}{\mu_i} f(h_i)]$$

We conclude that $\Delta\Pi_i > 0$ if and only if:

$$cf(h_j + h_i) > \frac{\rho(\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i})}{1+q} \frac{1}{\mu_i} f(h_i) \qquad \text{(vi)}$$

Given Proposition 2, $\exists \bar{\rho}_2$ such that $\sum_{l \in N_j} \frac{1}{\mu_l} < 1 + \frac{q+1}{\rho}$ for any $\rho < \bar{\rho}_2$ in a PNE. Given that $\sum_{l \in N_j} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$, we can write that for any $\rho < \bar{\rho}_2$, $\frac{\rho(\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i})}{1+q} \in [1 + \frac{\rho}{1+q} \frac{1}{\mu_i}, 1 + \frac{\rho}{1+q}(1 + \frac{1}{\mu_i}))$. So, $lim_{\rho \to 0} \frac{\rho(\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i})}{1+q} = 1$. On the other hand, from Lemma 2 we know that $cf(h_j + h_i) > f(h_i)$ for $f(\cdot)$ linear or convex and $c \geq \frac{1}{2}$. Following the definition of limit, we conclude that for a linear or convex $f(\cdot)$ and for any $\varepsilon > 0$, there always exists a $\rho' \leq \bar{\rho}_2$ such that $|\frac{\rho(\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i})}{1+q} - 1| < \varepsilon$, $\forall \rho < \rho'$. Therefore, given that $\mu_i \geq 1$, we can state that, for any vector h, and in particular for any pair (h_j, h_i) we can find a sufficiently low value of ρ (say ρ') such that $\frac{\rho(\sum_{l \in N_i} \frac{1}{\mu_l} + \frac{1}{\mu_{i+1}})}{1+q}$ is sufficiently close to 1 to hold condition (vi) for any $\rho < \rho'$ (when $f(\cdot)$ is linear or convex).

- $k \neq i$. Let us assume the extreme (and less favorable) case in which $o_k < 1$ and $\sum_{l \in N_j} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$. The marginal payoff derived from adding the link g_{ij} to the network is positive when:

$$f(h_j + h_i) > \frac{(\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i})\rho}{1+q} \frac{1}{\mu_i} f(h_i + h_k) \tag{vii}$$

From the previous point we know that for $\rho < \bar{\rho}_2$, $lim_{\rho \to 0} \frac{\rho(\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i})}{1+q} = 1$. Given that in case (b) $h_j > h_k$ and that $\mu_i \geq 1$, following the definition of limit we conclude that there always exists a $\rho'' < \bar{\rho}_2$ such that $\frac{\rho(\sum_{l \in N_i} \frac{1}{\mu_l} + \frac{1}{\mu_{i+1}})}{1+q}$ is sufficiently close to 1 to hold condition (vii) for any $\rho < \rho''$.

(c) There is a third case in which i has a unique link to k and $h_k > h_j$. For any $\rho < \bar{\rho}_1$, no other node has more than one link. Since g is a PNE, this implies that if $g_{ik} = 1$, either $g_{ki} = 1$ and then k is in case (b) or $\exists l \in N$ such that $g^s_{lk} = 1$ and $g_{lk} = 0$; otherwise, agent k would have incentives to cut some in-degree link off (from the definition of g^s). If $\exists l \in N$ such that $g^s_{lk} = 1$ and $g_{lk} = 0$, notice that $g_{lr} = 1$ for some $r \in N$. If $h_r < h_k$ agent l is in case (b). If $h_r > h_k$ we are able to repeat the same argument as before. Since n is finite, we eventually reach an iteration in which some player would be in case (b). Consequently, if a player is in case (c), for a sufficiently low ρ there must exist a different player in case (b).

But for the link g_{ij} to be formed, node j must agree. If $o_j \leq 1$ after the deviation, player j's marginal payoff will be positive. On the other hand, let us consider the case in which $\sum_{l \in N_j} \frac{1}{\mu_l} \geq \frac{q+1}{\rho}$ (the case in which $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ is analogous). Player j's marginal payoff for forming $g_{i,j}$ is:

$$\Delta\Pi_j = \frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l} + \frac{1}{\mu_i + 1}} [\sum_{l \in N_j \setminus \{j\}} c\frac{1}{\mu_l} f(h_l + h_j) + g_{jj} \frac{1}{\mu_j} f(h_j) + \frac{1}{\mu_i + 1} c f(h_i + h_j)] - \frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}} [\sum_{l \in N_j \setminus \{j\}} c\frac{1}{\mu_l} f(h_l + h_j) + g_{jj} \frac{1}{\mu_j} f(h_j)]$$

After simple algebra we conclude that $\Delta\Pi_j > 0$ if and only if:

$$cf(h_i + h_j) > \frac{\frac{1}{\mu_i + 1}}{\sum_{l \in N_j} \frac{1}{\mu_l}} [\sum_{l \in N_j \setminus \{j\}} c\frac{1}{\mu_l} f(h_l + h_j) + g_{jj} \frac{1}{\mu_j} f(h_j)] \tag{viii}$$

where the RHS can be interpreted as $\frac{1}{\mu_i + 1}$ times the average productivity of the ideas stored in the queue of j. Given that agent j receives some link in g^s, we conclude that j will have a relatively high level of talent. Consequently, we can say that there should be a ρ^o such that for any $\rho < \rho^o$ the average productivity of the ideas of the queue of i depends positively on ρ (and, in consequence, it depends negatively on $\sum_{l \in N_j} \frac{1}{\mu_l}$). Therefore, since $\mu_i + 1 \geq 1$, it should exist a ρ''' such that for any $\rho < \rho'''$ condition (viii) holds.

Given that $\rho' \leq \bar{\rho}_2$ and $\rho'' \leq \bar{\rho}_2 \leq \bar{\rho}_1$, if we define $\hat{\rho}_1$ as $min(\rho', \rho'', \rho''')$ the initial claim is proved.
■

Proof of Proposition 5. For g^s to be a PNE network, it should be robust to each of the three deviations of the following list. We show that, for a sufficiently low ρ, g^s is robust to each of them. Let $g^s_{ij} = 1$

a Agent i changes the destination of her open ideas from j to herself (by definition of g^s, $h_i < h_j$). We claim that agent i's marginal payoff obtained from this deviation is negative for a sufficiently low value of ρ, and therefore, network g^s is robust to such a deviation for a sufficiently low ρ. To show it, let us consider the extreme (and less favorable) case in which researcher j holds $\sum_{l\in N_j}\frac{1}{\mu_l} \geq \frac{1+q}{\rho}$ and agent i holds $\sum_{l\in N_i}\frac{1}{\mu_l}+1 < \frac{1+q}{\rho}$. In that case, i's marginal payoff will be negative when the following condition holds:

$$f(h_i) < \frac{1+q}{\rho\sum_{l\in N_j}\frac{1}{\mu_l}} cf(h_i+h_j) \tag{ix}$$

Given that j receives some link in g^s (then, she has a relatively high level of talent), we can use Corollary 1 to state that $\frac{q+1}{\rho}-1 \leq \sum_{l\in N_j(g^s)}\frac{1}{\mu_l} < 1+\frac{q+1}{\rho}$ for a $\rho < \bar{\rho}_2$. After simple algebra, we observe that it is equivalent to say that $\frac{1+q}{1+q+\rho} < \frac{1+q}{\rho\sum_{l\in N_j}\frac{1}{\mu_l}} \leq \frac{1+q}{1+q-\rho}$. So, we conclude that for $\rho < \bar{\rho}_2$, $lim_{\rho\to 0}\frac{1+q}{\rho\sum_{l\in N_j}\frac{1}{\mu_l}} = 1$. On the other hand, from Lemma 2 we know that $cf(h_j+h_i) > f(h_i)$ for $f(\cdot)$ linear or convex, $h_i < h_j$ and $c \geq \frac{1}{2}$. Following the definition of limit, we can say that for any $\varepsilon > 0$, there always exists a $\rho' < \bar{\rho}_2$ such that $|\frac{1+q}{\rho\sum_{l\in N_j}\frac{1}{\mu_l}}-1| < \varepsilon$, $\forall\rho < \rho'$. Then, for any h vector, and in particular for any pair (h_i, h_j) we can always find a sufficiently low value of ρ (say ρ') such that condition (ix) is hold for any $\rho < \rho'$, when $f(\cdot)$ is linear or convex. In this case, agent i will not have incentives to deviate.

b Agent i deletes one (or more) in-degree link. Given Lemma 1, the agent i's marginal payoff for deviating will be negative. Thus, g^s is robust to such deviation.

c Agent i proposes an additional link to some agent k with $h_k > h_j$. Given Lemma 1, agent k will have a negative marginal payoff for accepting the link g_{ik}. Thus g^s is also robust to such deviation by definition.

d Node i proposes the formation of the additional link g_{ik} to some player k with $h_k < h_j$[14]. We claim that the agent i's marginal payoff obtained from this deviation is negative for a sufficiently low value of ρ, and therefore, network g^s is also robust to such a deviation for a sufficiently low ρ. Let us consider the extreme (and less favorable) case in which $\sum_{l\in N_k}\frac{1}{\mu_l}+\frac{1}{2} < \frac{1+q}{\rho}$ and $\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2} \geq \frac{1+q}{\rho}$. Researcher i's marginal utility obtained from that deviation would be:

$$\Delta\Pi_i = \frac{c}{\mu_i+1}[\frac{1}{\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2}}f(h_i+h_j)+\frac{\rho}{1+q}f(h_i+h_k)] - \frac{1}{\mu_i\sum_{l\in N_j}\frac{1}{\mu_l}}cf(h_i+h_j)$$

where $\mu_i = 1$. After simple algebra, we observe that $\Delta\Pi_i > 0$ if and only if:

$$f(h_i+h_k) > f(h_i+h_j)\frac{1+q}{\rho}\frac{\sum_{l\in N_j}\frac{1}{\mu_l}-1}{(\sum_{l\in N_j}\frac{1}{\mu_l}-\frac{1}{2})(\sum_{l\in N_j}\frac{1}{\mu_l})} \tag{x}$$

[14]Notice that k is not necessarily different from i. The case in which $k = i$ is considered in part (e).

We know that $h_k < h_j$. On the other hand, given that j has a relatively high level of talent, we can use Corollary 1 to state that $\frac{q+1}{\rho} - 1 \leq \sum_{l\in N_j(g^s)} \frac{1}{\mu_l} < 1 + \frac{q+1}{\rho}$ for $\rho < \bar{\rho}_2$. Then we conclude that, for $\rho < \bar{\rho}_2$, $lim_{\rho \to o} \frac{1+q}{\rho} \frac{\sum_{l\in N_j} \frac{1}{\mu_l} - 1}{(\sum_{l\in N_j} \frac{1}{\mu_l} - \frac{1}{2})(\sum_{l\in N_j} \frac{1}{\mu_l})} = 1$. Since $h_k < h_j$ and following the definition of limit, we conclude that for any vector h there always exists a sufficiently low value of ρ (say $\rho'' < \bar{\rho}_2$) such that condition (x) holds for any $\rho < \rho''$ and, as a consequence, g^s is robust to this deviation.

e Node i deviates by forming an additional link towards herself. We claim that the marginal payoff obtained from this deviation is negative for a sufficiently low value of ρ, and therefore, network g^s is also robust to such a deviation when ρ is sufficiently low. Let us consider the extreme (and less favorable) case in which $\sum_{l\in N_i} \frac{1}{\mu_l} + \frac{1}{2} < \frac{1+q}{\rho}$ and $\sum_{l\in N_j} \frac{1}{\mu_l} - \frac{1}{2} \geq \frac{1+q}{\rho}$. The marginal payoff obtained by i would be:

$$\Delta\Pi_i = \frac{1}{\mu_i + 1}[\frac{c}{\sum_{l\in N_j} \frac{1}{\mu_l} - \frac{1}{2}} f(h_i + h_j) + \frac{\rho}{1+q} f(h_i)] - \frac{c}{\mu_i} \frac{1}{\sum_{l\in N_j} \frac{1}{\mu_l}} f(h_i + h_j)$$

After simple algebra, we see that $\Delta\Pi_i > 0$ if and only if:

$$f(h_i) > cf(h_i + h_j) \frac{1+q}{\rho} \frac{\sum_{l\in N_j} \frac{1}{\mu_l} - 1}{(\sum_{l\in N_j} \frac{1}{\mu_l} - \frac{1}{2})(\sum_{l\in N_j} \frac{1}{\mu_l})} \quad \text{(xi)}$$

From the definition of g^s we know that $h_i < h_j$. Repeating the same arguments as before we conclude that $lim_{\rho \to o} \frac{1+q}{\rho} \frac{\sum_{l\in N_j} \frac{1}{\mu_l} - 1}{(\sum_{l\in N_j} \frac{1}{\mu_l} - \frac{1}{2})(\sum_{l\in N_j} \frac{1}{\mu_l})} = 1$. On the other hand, from Lemma 2 we know that $cf(h_j + h_i) > f(h_i)$ for $f(\cdot)$ linear or convex, $h_i < h_j$ and $c \geq \frac{1}{2}$. Following the definition of limit, we can say that for any $\varepsilon > 0$, there always exists a ρ''' ($\rho''' < \bar{\rho}_2$) such that $|\frac{1+q}{\rho} \frac{\sum_{l\in N_j} \frac{1}{\mu_l} - 1}{(\sum_{l\in N_j} \frac{1}{\mu_l} - \frac{1}{2})(\sum_{l\in N_j} \frac{1}{\mu_l})} - 1| < \varepsilon, \forall \rho < \rho'''$. Then, for any h vector, and in particular for any pair (h_i, h_j) we can always find a sufficiently low value of ρ (say ρ''') such that condition (xi) is hold for any $\rho < \rho'''$, when $f(\cdot)$ is linear or convex. In this case agent i will not have incentives to deviate; so g^s is robust to this deviation.

Defining $\hat{\rho}_2$ as $min(\rho', \rho'', \rho''')$ the claim of the proposition is proved. ■

Proof of Proposition 6. Let us divide the proof in two steps. First, we want to show that in g^s if we substitute the link g_{ij} by another one, say g_{ik}, such that $h_j > h_k$, the aggregate payoff will decrease when ρ is sufficiently small. Given g^s, $\mu_r = 1$ and $\sum_{l\in N_r} \frac{1}{\mu_i} = \eta_r$, $\forall r \in N$. Following the definition of g^s, if $\eta_r > 0$ then η_r can take one of these two possible values, $\eta_r = \bar{\eta}$ or $\eta_r = \bar{\eta} - 1$. In the proof of lemma 1 we show that:

$$\frac{q+1}{\rho} - 1 \leq \bar{\eta} - 1 < \frac{q+1}{\rho} \leq \bar{\eta} < \frac{q+1}{\rho} + 1$$

Let us assume the less favorable case (the one in which this marginal aggregate payoff is maximum) in which agent j has $\frac{q+1}{\rho} \leq \sum_{l\in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$ and $o_k < 1$ even after receiving the additional link. In such a case:

$$\triangle \sum_{i \in N} \Pi_i = \frac{\rho}{1+q}[\sum_{l \in N_j \setminus \{i,j\}} cf(h_l + h_j) + g_{jj} f(h_j)] + \frac{2\rho}{1+q} cf(h_i + h_k) -$$
$$[\frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}}[\sum_{l \in N_j \setminus \{j\}} cf(h_l + h_j) + g_{jj} f(h_j)] + \frac{1}{\sum_{l \in N_j} \frac{1}{\mu_l}} cf(h_i + h_j)]$$

After simple algebra $\triangle \sum_{i \in N} \Pi_i > 0$ if and only if:

$$(\frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} - 1)[\sum_{l \in N_j \setminus \{i,j\}} cf(h_l + h_j) + g_{jj} f(h_j)] > 2[cf(h_i + h_j) - \frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} cf(h_i + h_k)]$$

Since $\frac{q+1}{\rho} \leq \sum_{l \in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$, $\frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} \in [1, 1 + \frac{\rho}{1+q})$. Then we can say that $lim_{\rho \to 0} \frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} = 1$. Given that $h_j > h_k$, we can say that $\exists \rho'$ such that the RHS will be higher than certain ε (> 0) for any $\rho < \rho'$. On the other hand, $lim_{\rho \to 0}(\frac{\rho \sum_{l \in N_j} \frac{1}{\mu_l}}{1+q} - 1) = 0$. Consequently, for the previous $\varepsilon > 0$, $\exists \rho''$ such that the LHS will be lower than ε for any $\rho < \rho''$. Then, we can always find a value of ρ (say ρ''') such that the last inequality will not hold for any h and for any $\rho < \rho'''$.

Second, we want to show that in g^s if we substitute the link g_{ij} by another one, say g_{ik}, such that $h_j < h_k$, the aggregate payoff will increase when ρ is sufficiently small. Let us assume the less favorable case (the one in which this marginal aggregate payoff is minimum) in which agent j has $\frac{q+1}{\rho} - 1 \leq \sum_{l \in N_j} \frac{1}{\mu_l} < \frac{q+1}{\rho}$ and $o_k \geq 1$. In such a case:

$$\triangle \sum_{i \in N} \Pi_i = \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l} + 1}(\sum_{l \in N_k} cf(h_l + h_k) + g_{kk} f(h_k) + cf(h_i + h_k)) + \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l} + 1} cf(h_i + h_k)$$
$$+ \frac{\rho}{1+q} \sum_{l \in N_j \setminus i} cf(h_l + h_j) - \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l}}(\sum_{l \in N_k} cf(h_l + h_k) + g_{kk} f(h_k))$$
$$- \frac{\rho}{1+q} cf(h_i + h_j) - \frac{\rho}{1+q} \sum_{l \in N_j} cf(h_l + h_j)$$

After simple algebra $\triangle \sum_{i \in N} \Pi_i > 0$ if and only if:

$$\frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l}}(\sum_{l \in N_k} cf(h_l + h_k) + g_{kk} f(h_k)) < 2(cf(h_i + h_k) - \frac{\rho(\sum_{l \in N_k} \frac{1}{\mu_l} + 1)}{1+q} cf(h_i + h_j))$$

Since $o_k \geq 1$, we can say that $\frac{q+1}{\rho} \leq \sum_{l \in N_k} \frac{1}{\mu_l} < \frac{q+1}{\rho} + 1$. Then $\frac{\rho(\sum_{l \in N_k} \frac{1}{\mu_l} + 1)}{1+q} \in [1 + \frac{\rho}{1+q}, 1 + \frac{2\rho}{1+q})$. Consequently, $lim_{\rho \to 0} \frac{\rho(\sum_{l \in N_k} \frac{1}{\mu_l} + 1)}{1+q} = 1$. Since $h_j < h_k$, we can say that $\exists \rho^{iv}$ such that the RHS will be higher than a certain $\varepsilon > 0$ for any $\rho < \rho^{iv}$. On the other hand, $lim_{\rho \to 0} \frac{1}{\sum_{l \in N_k} \frac{1}{\mu_l}} = 0$. Then, for the previous $\varepsilon > 0$, $\exists \rho^v$ such that the LHS will be lower than ε for any $\rho < \rho^v$. Then, we can always find a value of ρ (say ρ^{vi}) such that the last inequality will hold for any h and any $\rho < \rho^{vi}$.

Defining ρ^* as $min(\rho''', \rho^{vi})$ the statement of Proposition 6 is proved. ■